**Green’s Theorem Review**

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

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**Review**

(1) We have two notations for the line integral of a vector field on the plane:

\[ \int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{and} \quad \int_C F_1 \, dx + F_2 \, dy. \]

(2) \(\partial \mathcal{D}\) denotes the boundary of \(\mathcal{D}\) with its boundary orientation.

(3) Green’s Theorem:

\[
\oint_{\partial \mathcal{D}} F_1 \, dx + F_2 \, dy = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA
\]

or

\[
\oint_{\partial \mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}_z(\mathbf{F}) \, dA.
\]

(4) Formulas for the area of a region \(\mathcal{D}\) enclosed by \(\mathcal{C}\):

\[
\text{Area}(\mathcal{D}) = \oint_{\mathcal{C}} x \, dy = \oint_{\mathcal{C}} -y \, dx = \frac{1}{2} \oint_{\mathcal{C}} x \, dy - y \, dx.
\]

(5) The quantity

\[
\text{curl}_z(\mathbf{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}
\]

is interpreted as *circulation per unit area*. If \(\mathcal{D}\) is a small domain with boundary \(\mathcal{C}\), then for any \(P \in \mathcal{D}\),

\[
\oint_{\mathcal{C}} F_1 \, dx + F_2 \, dy \approx \text{curl}_z(\mathbf{F})(P) \cdot \text{Area}(\mathcal{D}).
\]

(6) Vector Form of Green’s Theorem:

\[
\oint_{\partial \mathcal{D}} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div}(\mathbf{F}) \, dA.
\]

The right-hand side of the identity above is called the *flux* of \(\mathbf{F}\) out of the unit circle.
**Problems**

(1) Use Green’s Theorem to evaluate the line integral. Orient the curve counterclockwise.

(a) \[ \oint_C y^2\, dx + x^2\, dy, \text{ where } C \text{ is the boundary of} \]
the unit square \(0 \leq x \leq 1, 0 \leq y \leq 1.\)

(b) \[ \oint_C e^{2x+y}\, dx + e^{-y}\, dy, \text{ where } C \text{ is the triangle with vertices (0,0), (1,0), and (1,1).} \]

(2) Evaluate \( I = \int_C (\sin x + y)\, dx + (3x + y)\, dy \) for the nonclosed path \(ABCD\) in Figure 1.

(3) Let \( C_R \) be the circle of radius \( R \) centered at the origin. Use the general form of Green’s Theorem to determine \( \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}, \) where \( \mathbf{F} \) is a vector field such that \( \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 9 \) and \( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = x^2 + y^2 \) for \((x,y)\) in the annulus \( 1 \leq x^2 + y^2 \leq 4.\)

(4) Referring to Figure 2, suppose that

\[ \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 3\pi \quad \text{and} \quad \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4\pi. \]

Use Green’s Theorem to determine the circulation of \( \mathbf{F} \) around \( C_1, \) assuming that \( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 9 \) on the shaded region.

(5) Compute the flux of \( \mathbf{F}(x,y) = (xy^2 + 2x, x^2y - 2y) \) across the simple closed curve that is the boundary of the half-disk given by \( x^2 + y^2 \leq 9, \, y \geq 0.\)