(1) A surface $S$ is oriented if a continuously varying unit normal vector $n(P)$ is specified at each point on $S$. This distinguishes an “outward” direction on the surface.

(2) The integral of a vector field $F$ over an oriented surface $S$ is defined as the integral of the normal component $F \cdot n$ over $S$.

(3) Vector surface integrals are computed using the formula

$$\int\int_S F \cdot dS = \int\int_D F(G(u, v)) \cdot N(u, v) \, du \, dv$$

Here, $G(u, v)$ is a parametrization of $S$ such that $N(u, v) = T_u \times T_v$ points in the direction of the unit normal vector specified by the orientation.

(4) The surface integral of a vector field $F$ over $S$ is also called the flux of $F$ through $G$. If $F$ is the velocity field of a fluid, then $\int\int_S F \cdot dS$ is the rate at which fluid flows through $S$ per unit time.
Problems

(1) Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given surface and vector field.

(a) $\mathbf{F} = \langle e^z, z, x \rangle$, $G(r, s) = (rs, r + s, r)$, $0 \leq r \leq 1$, $0 \leq s \leq 1$, oriented by $\mathbf{T}_r \times \mathbf{T}_s$.

(b) $\mathbf{F} = \langle z, z, x \rangle$, $z = 9 - x^2 - y^2$, $x \geq 0$, $y \geq 0$, $z \geq 0$, upward-pointing normal.

(2) The electric field due to a point charge located at the origin in $\mathbb{R}^3$ is

$$\mathbf{E} = k \frac{\mathbf{e}_r}{r^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and $k$ is a constant. Calculate the flux of $\mathbf{E}$ through the disk $D$ of radius 2 parallel to the $xy$-plane with center $(0, 0, 3)$.

(3) Let $S$ be the ellipsoid $\left(\frac{x}{\xi}\right)^2 + \left(\frac{y}{\eta}\right)^2 + \left(\frac{z}{\zeta}\right)^2 = 1$. Calculate the flux of $\mathbf{F} = z \mathbf{i}$ over the portion of $S$ where $x, y, z \leq 0$ with upward-pointing normal.

(4) Prove that if $S$ is the part of the graph $z = g(x, y)$ lying over a domain $D$ in the $xy$-plane, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3 \right) dx \, dy.$$