Review

(1) A *parametrized surface* is a surface $S$ whose points are described in the form

$$G(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where the *parameters* $u$ and $v$ vary in a domain $D$ in the $uv$-place.

(2) Tangent and normal vectors:

- **Tangent vectors:**
  $$T_u = \frac{\partial G}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}, \quad T_v = \frac{\partial G}{\partial v} = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}.$$  

- **Normal vector:**
  $$N = N(u, v) = T_u \times T_v.$$

(3) The quantity $\|N\|$ is an “area distortion factor”. If $D$ is a small region in the $uv$-plane and $S = G(D)$, then

$$\text{Area}(S) \approx \|N(u_0, v_0)\| \text{Area}(D),$$

where $(u_0, v_0)$ is *regular* at $(u, v)$ if $N(u, v) \neq 0$.

(4) Surface integrals and surface area:

$$\iint_S f(x, y, z) dS = \iint_D f(G(u, v)) \|N(u, v)\| du dv,$$

$$\text{Area}(S) = \iint_D \|N(u, v)\| du dv.$$

(5) Some standard parametrizations:

- **Cylinder of radius $R$ (z-axis as central axis):**
  $$G(\theta, z) = (R \cos \theta, R \sin \theta, z), \quad N = T_\theta \times T_z = R(\cos \theta, \sin \theta, 0), \quad dS = \|N\| d\theta dz = R d\theta dz.$$

- **Sphere of radius $R$, centered at the origin:**
  $$G(\theta, \phi) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi), \quad \text{Unit radial vector: } e_r = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi),$$

  $$\text{Outward normal: } N = T_\phi \times T_\theta = (R^2 \sin \phi) e_r, \quad dS = \|N\| d\phi d\theta = R^2 \sin \phi d\phi d\theta.$$

(6) Graph of $z = g(x, y)$:

$$G(x, y) = (x, y, g(x, y)), \quad N = T_x \times T_y = (-g_x, -g_y, 1), \quad dS = \|N\| dx dy = \sqrt{1 + g_x^2 + g_y^2} dx dy.$$
PROBLEMS

(1) Let $S = G(\mathcal{D})$, where $\mathcal{D} = \{(u, v) : u^2 + v^2 \leq 1, u \geq 0, v \geq 0\}$ and $G(u, v) = (2u + 1, u - v, 3u + v)$.

(a) Calculate the surface area of $S$. SOLUTION: $\frac{\pi \sqrt{6}}{2}$.

(b) Evaluate $\iint_S (x - y) dS$. Hint: use polar coordinates. SOLUTION: Integrate over $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$. The result is $\frac{8 + 3\pi}{\sqrt{6}}$.

(2) Calculate $\iint_S f(x, y, z) dS$ for the given surface and function.

(a) $G(u, v) = (u, v^3, u + v)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$; $f(x, y, z) = y$. SOLUTION: $\frac{19\sqrt{19} - 1}{108}$.

(b) Part of the unit sphere centered at the origin, where $x \geq 0$ and $|y| \leq x$; $f(x, y, z) = x$. SOLUTION: $\frac{\pi}{\sqrt{2}}$.

(3) Find the surface area of the part of the cone $x^2 + y^2 = z^2$ between the planes $z = 2$ and $z = 5$.
SOLUTION: Parametrize it by $\Phi(u, v) = (u \cos v, u \sin v, u)$, $\mathcal{D}: 0 \leq v \leq 2\pi, 2 \leq u \leq 5$. The normal vector $N$ will be $N(u, v) = (-u \cos v, -u \sin v, 1)$. Use that to conclude that the area is $21\sqrt{2}\pi$.

(4) Calculate $\iint_G x^2 z dS$, where $G$ is the cylinder (including the top and the bottom) $x^2 + y^2 = 4$, $0 \leq z \leq 3$.
SOLUTION: Compute the top, bottom and side contributions separately and add them up. The answer is $48\pi$. 