Rapid Review

(1) The limit of a product $f(x, y) = g(x)h(y)$ is a product of limits:

$$
\lim_{(x, y) \to (a, b)} f(x, y) = \left( \lim_{x \to a} g(x) \right) \left( \lim_{y \to b} h(y) \right).
$$

(2) A function $f$ of two variables is continuous at $P = (a, b)$ if

$$
\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b).
$$

(3) To prove that a limit does not exist, it suffices to show that the limits obtained along two different paths are not equal.

(4) The partial derivatives of $f(x, y)$ are defined as the limits

$$
f_x(a, b) = \frac{\partial f}{\partial x} \bigg|_{(a, b)} = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h},
$$

$$
f_y(a, b) = \frac{\partial f}{\partial y} \bigg|_{(a, b)} = \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k}.
$$

(5) Equation of the tangent plane to $z = f(x, y)$ at $(a, b)$:

$$
z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).
$$

(6) Linear approximation:

$$
f(a + \Delta x, b + \Delta y) \approx f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y
$$

$$
\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y.
$$

(7) The gradient of $f$ is the vector

$$
\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.
$$

(8) Chain rule for paths: $\frac{d}{dt} f(\mathbf{r}(t)) = \frac{df}{dx} \cdot \frac{d\mathbf{r}}{dt}$. \hfill (1)

(9) Formula for the directional derivative with respect to $\mathbf{u}$: $D_\mathbf{u}f(a, b) = \frac{df}{ds} \cdot \mathbf{u}$. \hfill (2)
Problems

1. Let \( f(x, y) = \frac{x^3 + y^3}{xy} \). Does \( \lim_{(x,y) \to (0,0)} f(x, y) \) exist? (Hint: set \( y = mx \) and show that the result depends on \( m \)).

2. Use any method to evaluate the limit or show that it does not exist
   (a) \( \lim_{(x,y) \to (0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} \).
   (b) \( \lim_{(x,y) \to (0,0)} \frac{x^2-y^2}{x^2+y^2} \).
   (c) \( \lim_{(x,y) \to (0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \).

3. Compute the first-order partial derivatives:
   (a) \( z = e^{-x^2-y^2} \).
   (b) \( w = \frac{x}{y+z} \).

4. The volume of a right-circular cone of radius \( r \) and height \( h \) is \( V = \frac{\pi}{3} r^2 h \). Use linear approximation to estimate the percentage change in volume of a right-circular cone of radius \( r = 40 \text{cm} \) if the height is increased from 40 to 41 cm.

5. Find the points on the graph of \( z = 3x^2 - 4y^2 \) at which the vector \( \mathbf{n} = \langle 3, 2, 2 \rangle \) is normal to the tangent plane.

6. Compute \( \frac{d}{dt} f(r(t)) \) at the given point:
   (a) \( f(x, y) = x^3 - 3xy, \, r(t) = \langle \cos t, \sin t \rangle, \, t = 0 \).
   (b) \( f(x, y) = \sin(xy), \, r(t) = \langle e^{2t}, e^{3t} \rangle, \, t = 0 \).

7. A bug located at (3, 9, 4) begins walking in a straight line toward (5, 7, 3). At what rate is the bug's temperature changing if the temperature is \( T(x, y, z) = xe^{y-z} \)? Units are in meters and degrees in Celsius.