Rapid Review

(1) The length $s$ of a path $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ is

$$s = \int_a^b \| \mathbf{r}'(t) \| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$ 

(2) Arc length function: $s(t) = \int_a^t \| \mathbf{r}'(u) \| du$.

(3) Speed is the derivative of distance traveled with respect to time:

$$v(t) = \frac{ds}{dt} = \| \mathbf{r}'(t) \|.$$ 

(4) $\mathbf{r}(s)$ is an arc length parametrization if $\| \mathbf{r}'(s) \| = 1$ for all $s$.

(5) If $\mathbf{r}(t)$ is any parametrization such that $\mathbf{r}'(t) \neq 0$ for all $t$, then

$$\mathbf{r}_1(s) = \mathbf{r}(g^{-1}(s))$$

is an arc length parametrization, where $t = g^{-1}(s)$ is the inverse function of the arc length function $s = g(t)$.

(6) The domain $D$ of a function $f(x_1, \ldots, x_n)$ is the set of $n$-tuples $(a_1, \ldots, a_n)$ in $\mathbb{R}^n$ for which the function is defined. For example, the domain of

$$f(x_1, \ldots, x_n) = \frac{1}{\| (x_1, \ldots, x_n) \|}$$

is $\mathbb{R}^n - \{(0, \ldots, 0)\}$. The range of $f$ is the set of values taken by $f$.

(7) The graph of a real-valued function $f$ is the subset of $\mathbb{R}^3$ of points $(a, b, f(a, b))$, for $(a, b)$ in the domain of $f$.

(8) A vertical trace is obtained by intersecting the graph with a vertical plane $x = a$ or $y = b$.

(9) A level curve is a curve in the $xy$-plane defined by an equation $f(x, y) = c$.

(10) The contour map shows the level curves $f(x, y) = c$ for equally spaced values of $c$. The spacing $m$ is called the contour interval.

(11) A level surface is a surface in the $xyz$-space defined by an equation $f(x, y, z) = c$. If $f$ represents temperature, we call the level surfaces isotherms.
PROBLEMS

(1) Compute the arc length of the following curves over the given interval:

(a) \( \mathbf{r}(t) = (\cos t, \sin t, t^{3/2}), \) \( 0 \leq t \leq 2\pi. \)

\textbf{Solution:} \( \frac{2}{27}((2 + 9\pi)^{3/2}\sqrt{2} - 4). \)

(b) \( \mathbf{r}(t) = (t, 4t^{3/2}, 2t^{3/2}), \) \( 0 \leq t \leq 3. \)

\textbf{Solution:} \( \frac{2}{135}(136^{3/2} - 1). \)

(2) Find the speed of \( \mathbf{r}(t) = (t, \ln t, (\ln t)^2) \) at \( t = 1. \)

\textbf{Solution:} \( \sqrt{2}. \)

(3) Find an arc length parametrization of the circle in the plane \( z = 9 \) with radius 4 and center \((1, 4, 9). \)

\textbf{Solution:} \( \mathbf{r}_1(s) = (1 + 4 \cos (s/4), 4 + 4 \sin (s/4), 9). \)

(4) Sketch the contour map of \( f(x, y) = x^2 + y^2 \) with level curves \( c = 0, 4, 8, 12, 16. \)

(5) Draw a contour map of \( f(x, y) = xy \) with an appropriate contour interval, showing at least 4 curves.

(6) Let the temperature in 3-space be given by \( T(x, y, z) = x^2 + y^2 - z^2. \) Draw isotherms corresponding to temperatures \( T = -2, 0, 2. \)

(7) Let the temperature in 3-space be given by \( T(x, y, z) = x^2 + y^2 - z. \) Draw isotherms corresponding to temperatures \( T = -1, 0, 1. \)