Roots of polynomials

(a) Suppose that $f''$ is continuous at every $x$ in $[a, b]$. Show that if $f''(x) > 0$ for all $x$ in $[a, b]$ then $f'(x)$ has at most one zero in $[a, b]$. What if $f'' < 0$ on $[a, b]$ instead?

(b) Show that the polynomial $f(x) = x^4 + 3x + 1$ has exactly one zero on the interval $[-2, -1]$.

(c) Show that a cubic (degree 3) polynomial can have at most 3 zeros.

Marathon runner

A marathoner ran the 26.2-mile New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph, assuming the initial and final speeds are zero.

Extreme values

Consider the function $f(x) = |x^3 - 9x|$.

(a) Does the function $f$ obtain its absolute maximum and minimum on the interval $[-2.5, 2]$? If yes, what are they?

(b) Same question(s) on the interval $[-5, 5]$.

(c) Same question on the whole real line: $(-\infty, \infty)$.

Changing angle

Point $B$ moves from point $A$ to point $C$ at 2 cm / sec in the accompanying diagram. At what rate is $\theta$ changing when $x = 4$ cm?
Cube’s changing edges

The volume of a cube is increasing at the rate of 1200 cm$^3$/min at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

Motion of a particle

The position at time $t \geq 0$ of a particle moving along a coordinate line is

$$s = 10 \cos(t + \pi/4).$$

(a) What is the particle’s starting position?
(b) What are the points farthest to the left and right of the origin reached by the particle?
(c) Find the particle’s velocity and acceleration at the points in part (b)?
(d) When does the particle first reach the origin? What are its velocity, speed and acceleration then?

Linearisation

Determine the linearisations of the following functions at the given points

(a) $f(x) = \tan x$ at $a = \pi$.
(b) $f(x) = \sqrt[3]{x}$ at $a = -8$.
(c) $f(x) = \sin^{-1} x$ at $a = \pi/12$.

Use part (b) to approximate $\sqrt[3]{-8.1}$.

Parallel tangents

Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the $x$-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?