Math 1910 November 02, 2017

ONE-PAGE REVIEW

(1) There are three numerical approximations to \( \int_{a}^{b} f(x) \, dx \):

(a) The midpoint rule: 
\[ M_N = \Delta x \left( f(c_1) + \ldots + f(c_N) \right), \quad c_j = a + \left( j + \frac{1}{2} \right) \Delta x. \]

(b) The trapezoid rule: 
\[ T_N = \frac{1}{2} \Delta x \left( y_0 + 2y_1 + 2y_2 + \ldots + 2y_{N-1} + y_N \right) \]

(c) Simpson’s rule: 
\[ S_N = \frac{1}{3} \Delta x \left( y_0 + 4y_1 + 2y_2 + \ldots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N \right) \]

(2) The arc length of \( f(x) \) on the interval \([a, b]\) is
\[ \int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx. \]

(3) The surface area of the surface obtained by rotating the graph of \( f(x) \) around the x-axis for \( a \leq x \leq b \) is
\[ 2\pi \int_{a}^{b} f(x) \sqrt{1 + f'(x)^2} \, dx. \]

(4) The \( n \)-th Taylor Polynomial centered at \( x = a \) for the function \( f \) is
\[ T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \]

(5) The error for the \( n \)-th Taylor Polynomial is
\[ |T_n(x) - f(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}, \]
where \( K \) is the maximum of \( |f^{(n+1)}(u)| \) over all \( u \) between \( a \) and \( x \).

(6) Taylor’s Theorem says that
\[ R_n(x) = T_n(x) - f(x) = \frac{1}{n!} \int_{a}^{x} (x-u)^n f^{(n+1)}(u) \, du. \]
PROBLEMS

(1) Find the $T_4$ approximation for $\int_0^4 \sqrt{x} \, dx$.

**SOLUTION:** Let $f(x) = \sqrt{x}$. We divide $[0,4]$ into 4 subintervals of width

$$\Delta x = \frac{4 - 0}{4} = 1,$$

with endpoints 0, 1, 2, 3, 4. With this data, we get

$$T_4 = \frac{1}{2} \Delta x \left( \sqrt{0} + 2 \sqrt{1} + 2 \sqrt{2} + 2 \sqrt{3} + \sqrt{4} \right) \approx 5.14626.$$  

(2) State whether $M_{10}$ underestimates or overestimates $\int_1^4 \ln(x) \, dx$.

**SOLUTION:** Let $f(x) = \ln(x)$. Then $f'(x) = \frac{1}{x}$ and

$$f''(x) = -\frac{1}{x^2} < 0$$

on the interval $[1,4]$, so $f(x)$ is concave down. Therefore, the midpoint rule overestimates the integral.

(3) For the curve curve $y = \ln(\cos x)$ over the interval $[0, \pi/4]$, set up an integral to calculate:

(a) the arc length.

**SOLUTION:** First, calculate

$$1 + (y')^2 = 1 + \tan^2(x) = \sec^2(x),$$

so the arc length is

$$\int_0^{\pi/4} \sqrt{1 + (y')^2} \, dx = \int_0^{\pi/4} \sec^2(x) \, dx = \int_0^{\pi/4} \sec(x) \, dx = \ln |\sec(x) + \tan(x)| \bigg|_0^{\pi/4} = \ln \sqrt{2} + 1.$$

(b) the surface area when rotated around the $x$-axis.

**SOLUTION:** As in the previous part, we have

$$1 + (y')^2 = \sec^2(x)$$

Therefore, plug into the arc length formula

$$\text{Surface Area} = 2\pi \int_0^{\pi/4} y \sqrt{1 + (y')^2} = 2\pi \int_0^{\pi/4} \ln(\cos(x)) \sec(x) \, dx.$$  

(4) Approximate the arc length of the curve $y = \sin(x)$ over the interval $[0, \pi/2]$ using the midpoint approximation $M_8$.

**SOLUTION:** Since $y = \sin(x)$, we have

$$1 + (y')^2 = 1 + \cos^2(x)$$
Therefore, $\sqrt{1+(y')^2} = \sqrt{1+\cos^2(x)}$, and the arc length over $[0, \pi/2]$ is

$$\int_0^{\pi/2} \sqrt{1+\cos^2(x)} \, dx.$$ 

Let $f(x) = \sqrt{1+\cos^2(x)}$. $M_8$ is the midpoint approximation with eight subdivisions. So

$$\Delta x = \frac{\pi/2 - 0}{8} = \frac{\pi}{16},$$

$$x_i = 0 + (i - \frac{1}{2})\Delta x \quad \text{for } i = 1, 2, \ldots, 8$$

$$y_i = f\left((i - \frac{1}{2})\Delta x\right)$$

$$M_8 = \sum_{i=1}^{8} y_i\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_8)\Delta x$$

<table>
<thead>
<tr>
<th>i</th>
<th>$x_i$</th>
<th>$f(x_i) = y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>1.41081</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>1.3841</td>
</tr>
<tr>
<td>3</td>
<td>2.5</td>
<td>1.3333</td>
</tr>
<tr>
<td>4</td>
<td>3.5</td>
<td>1.26394</td>
</tr>
<tr>
<td>5</td>
<td>4.5</td>
<td>1.18425</td>
</tr>
<tr>
<td>6</td>
<td>5.5</td>
<td>1.10554</td>
</tr>
<tr>
<td>7</td>
<td>6.5</td>
<td>1.04128</td>
</tr>
<tr>
<td>8</td>
<td>7.5</td>
<td>1.00479</td>
</tr>
</tbody>
</table>

The final answer is that the arc length is approximately $1.9101$.

(5) Find the Taylor polynomials $T_2(x)$ and $T_3(x)$ for $f(x) = \frac{1}{\sqrt{1+x}}$ centered at $a = 1$.

**SOLUTION:** We need to take a few derivatives, and then plug in $a = 1$ to each one.

<table>
<thead>
<tr>
<th>n</th>
<th>n-th derivative $f^{(n)}(x)$</th>
<th>$f^{(n)}(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$f(x) = \frac{1}{\sqrt{1+x}}$</td>
<td>$f(1) = 1/2$</td>
</tr>
<tr>
<td>1</td>
<td>$f'(x) = \frac{-1}{(1+x)^{3/2}}$</td>
<td>$f'(1) = -1/4$</td>
</tr>
<tr>
<td>2</td>
<td>$f''(x) = \frac{2}{(1+x)^{5/2}}$</td>
<td>$f''(1) = 1/4$</td>
</tr>
<tr>
<td>3</td>
<td>$f'''(x) = \frac{-6}{(1+x)^{7/2}}$</td>
<td>$f'''(1) = -3/8$</td>
</tr>
</tbody>
</table>

Then plug these values into the formula for the Taylor polynomial.

$$T_2(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8}$$

$$T_3(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{16}$$

(6) Find $n$ such that $|T_n(1.3) - \sqrt{1.3}| \leq 10^{-6}$, where $T_n$ is the Taylor polynomial for $\sqrt{x}$ at $a = 1$. 

3
SOLUTION: By the error formula, we have that

$$|T_n(1.3) - \sqrt{1.3}| \leq \frac{K_{n+1}(1.3 - 1)^{n+1}}{(n+1)!}$$

So we just need to find $n$ such that

$$\frac{K_{n+1}(0.3)^{n+1}}{(n+1)!} < 10^{-6},$$

where $K_{n+1}$ is the maximum value of the $(n+1)$-st derivative of $f(x) = \sqrt{x}$ between 1 and 1.3. Since $f^{(n+1)}(x)$ is the $(n+1)$-st derivative of $\sqrt{x}$, and this always has $x$ in the denominator for any $n \geq 0$, this maximum will always occur at $x = 1$. Therefore, in this case,

$$K_{n+1} = |f^{(n+1)}(1)|.$$  

So we just need to find $n$ such that

$$\frac{|f^{(n+1)}(1)|(0.3)^{n+1}}{(n+1)!} < 10^{-6}.$$ 

The hard part is finding a pattern for the $n$-th derivative of $\sqrt{x}$, but that’s not strictly necessary, although possible. If you keep taking derivatives of $\sqrt{x}$ and plugging into the formula, you find that this is valid for $n \geq 7$.

Alternatively, the general formula for the $n$-th derivative of $\sqrt{x}$ is

$$f^{(n)}(x) = (-1)^{n+1} \cdot 3 \cdot 5 \cdots (2n - 3) \frac{(2n - 3) \cdots (2)}{2^n} x^{-\frac{2n - 3}{2}}$$

Then you can plug this in to the previous formula.