§6.1: AREAS,
§6.2: VOLUMES,
§6.3: REVOLUTION

Math 1910 September 19, 2017

Add similar triangles review!

**ONE PAGE REVIEW**

(1) The graph of \( x = f(y) \) is the graph of \( y = f(x) \) reflected across the line \( y = x \). \( \text{(1)} \)

(2) The area between \( f(x) \) and \( g(x) \) from \( a \) to \( b \) is \( \int_{a}^{b} (y_{\text{top}} - y_{\text{bottom}}) \, dx \). \( \text{(2)} \)

(3) The **average value** of \( f(x) \) over the interval \([a, b]\) is \( \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \). \( \text{(3)} \)

(4) The **Mean Value Theorem for Integrals** says that if \( f \) is continuous on \([a, b]\) with mean value \( M \), then there is some \( c \in [a, b] \) such that \( f(c) = M \). \( \text{(4)} \)

(5) If a shape has cross-sectional area \( A(y) \) and height extends from \( y = a \) to \( y = b \), then it’s volume is \( \int_{a}^{b} A(y) \, dy \). \( \text{(5)} \)

(6) **Cavilieri’s Principle** says if two solids have equal cross-sectional areas, then they also have equal volumes. \( \text{(6)} \)

(7) **The Disk Method**: If \( f(x) \geq 0 \) on \([a, b]\), then the solid obtained by rotating the region under the graph around the \( x \)-axis has volume \( \int_{a}^{b} \pi f(x)^{2} \, dx \). \( \text{(7)} \)

(8) **The Washer Method**: If \( f(x) \geq g(x) \geq 0 \) on \([a, b]\), then the solid obtained by rotating the region between \( f(x) \) and \( g(x) \) around the \( x \)-axis has volume \( \int_{a}^{b} \pi (f(x)^{2} - g(x)^{2}) \, dx \). \( \text{(8)} \)
PROBLEMS

(1) Sketch the region enclosed by the curves and set up an integral to compute it’s area, but do not evaluate.

(a) \( y = 4 - x^2, \ y = x^2 - 4 \)

SOLUTION: Setting \( 4 - x^2 = x^2 - 4 \) yields \( 2x^2 = 8 \) or \( x^2 = 4 \). Thus, the curves \( y = 4 - x^2 \) and \( y = x^2 - 4 \) intersect at \( x = \pm 2 \). From the figure below, we see that \( y = 4 - x^2 \) lies above \( y = x^2 - 4 \) over the interval \([-2, 2]\); hence, the area of the region enclosed by the curves is

\[
\int_{-2}^{2} \left( (4 - x^2) - (x^2 - 4) \right) \, dx = \int_{-2}^{2} (8 - 2x^2) \, dx = \left( 8x - \frac{2}{3}x^3 \right) \bigg|_{-2}^{2} = \frac{64}{3}
\]

(b) \( y = x^2 - 6, \ y = 6 - x^3, \ x = 0 \)

SOLUTION: Setting \( x^2 - 6 = 6 - x^3 \) yields

\[
0 = x^3 + x^2 - 12 = (x - 2)(x^2 + 3x + 6),
\]

so the curves \( y = x^2 - 6 \) and \( y = 6 - x^3 \) intersect at \( x = 2 \). Using the graph shown below, we see that \( y = 6 - x^3 \) lies above \( y = x^2 - 6 \) over the interval \([0, 2]\); hence, the area of the region enclosed by these curves and the \( y \)-axis is

\[
\int_{0}^{2} \left( (6 - x^3) - (x^2 - 6) \right) \, dx = \int_{0}^{2} (-x^3 - x^2 + 12) \, dx = \left( -\frac{1}{4}x^4 - \frac{1}{3}x^3 + 12x \right) \bigg|_{0}^{2} = \frac{52}{3}
\]

(c) \( y = x\sqrt{x-2}, \ y = -x\sqrt{x-2}, \ x = 4 \)

SOLUTION: Note that \( y = x\sqrt{x-2} \) and \( y = -x\sqrt{x-2} \) are the upper and lower branches, respectively, of the curve \( y^2 = x^2(x - 2) \). The area enclosed by this curve and the vertical line \( x = 4 \) is

\[
\int_{2}^{4} \left( x\sqrt{x-2} - (-x\sqrt{x-2}) \right) \, dx = \int_{2}^{4} 2x\sqrt{x-2} \, dx.
\]

Substitute \( u = x - 2 \). Then \( du = dx \), \( x = u + 2 \) and

\[
\int_{2}^{4} 2x\sqrt{x-2} \, dx = \int_{0}^{2} 2(u+2)\sqrt{u} \, du = \int_{0}^{2} \left( 2u^{3/2} + 4u^{1/2} \right) \, du = \left( \frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2} \right) \bigg|_{0}^{2} = \frac{128\sqrt{2}}{15}.
\]
(d) \( x = 2y, x + 1 = (y - 1)^2 \)

**Solution:** Setting \( 2y = (y - 1)^2 - 1 \) yields \( 0 = y^2 - 4y = y(y - 4) \), so the two curves intersect at \( y = 0 \) and \( y = 4 \). From the graph below, we see that \( x = 2y \) lies to the right of \( x + 1 = (y - 1)^2 \) over the interval \([0, 4]\) along the \( y \)-axis. Thus, the area of the region enclosed by the two curves is

\[
\int_0^4 \left(2y - ((y - 1)^2 - 1)\right) \, dy = \int_0^4 \left(4y - y^2\right) \, dy = \left(2y^2 - \frac{1}{3}y^3\right) \bigg|_0^4 = \frac{32}{3}.
\]

(e) \( y = \cos x, \, y = \cos(2x), \, x = 0, \, x = \frac{2\pi}{3} \)

**Solution:** From the graph below, we see that \( y = \cos x \) lies above \( y = \cos 2x \) over the interval \([0, \frac{2\pi}{3}]\). The area of the region enclosed by the two curves is therefore

\[
\int_0^{2\pi/3} (\cos x - \cos 2x) \, dx = \left(\sin x - \frac{1}{2} \sin 2x\right) \bigg|_0^{2\pi/3} = \frac{3\sqrt{3}}{4}.
\]
(2) Calculate the volume of a cylinder inclined at an angle \( \theta = \frac{\pi}{6} \) with height 10 and base of radius 4.

\[ \text{SOLUTION: By Cavalieri’s Principle, the volume of this thing is the same as the volume of a regular cylinder of height 10. So the volume is } \pi R^2 h = \pi (4)^2 (10) = 160\pi. \]

Alternatively, the cross-sectional area is at each y-value \( \pi (4)^2 = 16\pi \), so the volume is

\[ \int_0^{10} 16\pi \, dy = 160\pi. \]

(3) Calculate the volume of the ramp in the figure below in three ways by integrating the area of the cross sections:

(a) perpendicular to the x-axis.

\[ \text{SOLUTION: Cross sections perpendicular to the x-axis are rectangles of width 4 and height } 2 - \frac{1}{3}x. \]

The volume of the ramp is then

\[ \int_0^6 4\left(-\frac{1}{3}x + 2\right) \, dx = \left(\frac{2}{3}x^2 + 8x\right)
\]

\[ \left. \right|_0^6 = 24. \]

(b) perpendicular to the y-axis.

\[ \text{SOLUTION: Cross sections perpendicular to the y-axis are right triangles with legs of length 2 and 6. The volume of the ramp is then} \]

\[ \int_0^4 \left(\frac{1}{2} \cdot 2 \cdot 6\right) \, dy = (6y)
\]

\[ \left. \right|_0^4 = 24. \]

(c) perpendicular to the z-axis.

\[ \text{SOLUTION: Cross sections perpendicular to the z-axis are rectangles of length } 6 - 3z \text{ and width 4. The volume of the ramp is then} \]

\[ \int_0^2 4(-3(z - 2)) \, dz = (-6z^2 + 24z)
\]

\[ \left. \right|_0^2 = 24. \]
(4) Let $M$ be the average value of $f(x) = 2x^2$ on $[0, 2]$. Find a value $c$ such that $f(c) = M$.

**Solution:** First find the average value

$$M = \frac{1}{2-0} \int_0^2 2x^2 \, dx = \frac{2}{2} \int_0^2 x^2 \, dx = \frac{1}{3} x^3 \bigg|_0^2 = \frac{8}{3}.$$  

Then $M = f(c) = 2c^2 = \frac{8}{3}$ implies $c = \frac{2}{\sqrt{3}}$.

(5) Find the flow rate through a tube of radius 2 meters, if it’s fluid velocity at distance $r$ meters from the center is $v(r) = 4 - r^2$.

**Solution:** The flow rate is (in meters cubed per second)

$$2\pi \int_0^2 rv(r) \, dr = 2\pi \int_0^2 r(4 - r^2) \, dr = 2\pi \left(2r^2 - \frac{1}{3} r^4\right) \bigg|_0^2 = 8\pi.$$  

(6) Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis $a = 6$ and semiminor axis $b = 4$.

**Solution:** At each height $y$, the elliptical cross section has major axis $\frac{1}{2}(12 - y)$ and minor axis $\frac{1}{3}(12 - y)$. The cross-sectional area is then $\frac{\pi}{6}(12 - y)^2$, and the volume is

$$\int_0^{12} \frac{\pi}{6}(12 - y)^2 \, dy = -\frac{\pi}{18}(12 - y)^3 \bigg|_0^{12} = 96\pi.$$  

(7) Sketch the region enclosed by the curves, and determine the cross section perpendicular to the $x$-axis. Set up an integral for the volume of revolution obtained by rotating the region around the $x$-axis, but do not evaluate.
(a) \( y = x^2 + 2, \quad y = 10 - x^2 \).

**SOLUTION:** Setting \( x^2 + 2x = 10 - x^2 \) yields \( x = \pm 2 \). The region enclosed is the figure below.

When this is rotated about the \( x \)-axis, each cross section is a washer with outer radius \( R = 10 - x^2 \) and inner radius \( r = x^2 + 2 \). The volume of the solid of revolution is then

\[
\pi \int_{-2}^{2} \left((10 - x^2)^2 - (x^2 + 2)^2\right) \, dx
\]

(b) \( y = 16 - x, \quad y = 3x + 12, \quad x = -1 \).

**SOLUTION:** Setting \( 16 - x = 3x + 12 \) gives \( x = 1 \). The region enclosed is in the picture below.

When rotated about the \( x \)-axis, each cross section is a washer with outer radius \( R = 16 - x \) and inner radius \( r = 3x + 12 \). So the volume of the solid of revolution is

\[
\pi \int_{-1}^{1} \left((16 - x)^2 - (3x + 12)^2\right) \, dx.
\]

(c) \( y = \frac{1}{x}, \quad y = \frac{5}{2} - x \).

**SOLUTION:** Setting \( \frac{1}{x} = \frac{5}{2} - x \) yields

\[
0 = x^2 - \frac{5}{2}x + 1 = (x - 2)(x - \frac{1}{2}).
\]

So \( x = 2 \) and \( x = \frac{1}{2} \) are the two points of intersection. The region enclosed by the curves is in the
The cross sections are washers with outer radius \( R = \frac{5}{2} - x \) and inner radius \( r = \frac{1}{x} \). So the volume is

\[
\pi \int_{1/2}^{2} \left( \left( \frac{5}{2} - x \right)^2 - \frac{1}{x^2} \right) \, dx
\]

(d) \( y = \sec x, \ y = 0, \ x = -\frac{\pi}{4}, \ x = \frac{\pi}{4} \).

**SOLUTION:** The region in question is

When rotated around the \( x \)-axis, each cross section is a circular disk with radius \( R = \sec x \). The volume of this solid of revolution is

\[
\pi \int_{-\pi/4}^{\pi/4} (\sec x)^2 \, dx
\]

(8) A frustrum of a pyramid is a pyramid with its top cut off. Let \( V \) be the volume of a frustrum of height \( h \) whose base is a square of side \( a \) and whose top is a square of side \( b \) with \( a > b > 0 \).
(a) Show that if the frustrum were continued to a full pyramid (i.e. the top wasn’t cut off), it would have height $ha/(a-b)$.

**SOLUTION:** Let $H$ be the height of the full pyramid. Using similar triangles, we have the proportion $H/a = (H-h)/b$, which gives $H = ha/(a-b)$.

(b) Show that the cross sectional area at height $x$ is a square of side $(1/h)(a(h-x) + bx)$.

**SOLUTION:** Let $w$ denote the side length of the square cross-section at height $x$. By similar triangles, we have $a/H = w/H-x$. Substituting $H$ from part (a) gives

$$w = \frac{a(h-x) + bx}{h}.$$ 

(c) Show that $V = \frac{1}{3}h(a^2 + ab + b^2)$.

**SOLUTION:** The volume of the frustrum is

$$\int_0^h \left(\frac{1}{h}(a(h-x) + bx)\right)^2 \, dx = \frac{1}{h^2} \int_0^h (a^2(h-x)^2 + 2ab(h-x)x + b^2x^2) \, dx$$

$$= \frac{1}{h^2} \left( -\frac{a^2}{3}(h-x)^3 + abhx^2 - \frac{2}{3}abx^3 + \frac{1}{3}b^2x^3 \right) \bigg|_0^h$$

$$= \frac{h}{3} \left( a^2 + ab + b^2 \right)$$