§5.3: Indefinite Integrals
§5.4, §5.5: Fundamental Theorem
Math 1910

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One-page Review

1. $F$ is called an antiderivative of $f$ if $F'(x) = f(x)$. Any two antiderivatives of $f$ on an interval $(a, b)$ differ by a constant.

2. Fundamental Theorem of Calculus, Part I (FTC I): if $F(x)$ is an antiderivative for $f(x)$, then

$$
\int_a^b f(x) \, dx = F(b) - F(a)
$$

3. (a) $\int 0 \, dx = C$
   (b) $\int k \, dx = kx + C$
   (c) $\int cf(x) \, dx = c\int f(x) \, dx$
   (d) $\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$
   (e) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
   (f) $\int \sin x \, dx = -\cos x + C$
   (g) $\int \sec^2 x \, dx = \tan x + C$
   (h) $\int \sec x \tan x \, dx = \sec x + C$

4. To solve an initial value problem $\frac{dy}{dx} = f(x), y(x_0) = y_0$, first find the general antiderivative $y = F(x) + C$. Then determine $C$ using the initial condition $F(x_0) + C = y_0$.

5. The area function with lower limit $a$ is $A(x) = \int_a^x f(t) \, dt$.

6. Fundamental Theorem of Calculus, Part II (FTC II):

$$
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
$$

7. A consequence of FTC II is that every continuous function has an antiderivative.

8. Let $G(x) = \int_a^x f(t) \, dt$. Let $A(x) = \int_a^x f(t) \, dt$. Then

$$
\frac{d}{dx} G(x) = \frac{d}{dx} \int_a^x f(t) \, dt = \frac{d}{dx} A(g(x)) = A'(g(x))g'(x) = f(g(x))g'(x)
$$
PROBLEMS

(1) Evaluate the integral:

(a) \[ \int \cos x \, dx \]

**SOLUTION:** \[ \int \cos x \, dx = \sin x + C \]

(b) \[ \int \csc x \cot x \, dx \]

**SOLUTION:** \[ \int \csc x \cot x \, dx = -\csc x + C \]

(c) \[ \int \frac{3}{x^{3/2}} \, dx \]

**SOLUTION:** Since \( \frac{3}{x^{3/2}} \, dx = 3x^{-3/2} \), we get

\[ \int \frac{3}{x^{3/2}} \, dx = \int 3x^{-3/2} \, dx \]
\[ = 3 \left( -\frac{1}{(-1/2)} x^{-1/2} \right) + C \]
\[ = -6x^{-1/2} + C \]

(d) \[ \int_{-2}^{9} (10x^9 + 3x^5) \, dx \]

**SOLUTION:** 
\[ \int_{-2}^{9} (10x^9 + 3x^5) \, dx = \left( x^{10} + \frac{1}{2}x^6 \right) \bigg|_{-2}^{9} = (2^{10} + \frac{1}{2}2^6) - (2^{10} + \frac{1}{2}2^6) = 0 \]

(e) \[ \int_{0}^{4} \sqrt{x} \, dx \]

**SOLUTION:** 
\[ \int_{0}^{4} \sqrt{x} \, dx = \int_{0}^{4} x^{1/2} \, dx = \frac{2}{3}x^{3/2} \bigg|_{0}^{4} = \frac{2}{3}(4^{3/2}) - \frac{2}{3}(0)^{3/2} = \frac{16}{3} \]

(f) \[ \int_{\pi/4}^{3\pi/4} \sin \theta \, d\theta \]

**SOLUTION:** 
\[ \int_{\pi/4}^{3\pi/4} \sin \theta \, d\theta = -\cos \theta \bigg|_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \]

(g) \[ \int_{0}^{5} |x^2 - 4x + 3| \, dx \]

**SOLUTION:** Write the integral as a sum of integrals without absolute values and then apply FTC I.

\[ \int_{0}^{5} |x^2 - 4x + 3| \, dx = \int_{0}^{5} (x - 3)(x - 1) \, dx \]
\[ = \int_{0}^{1} (x^2 - 4x + 3) \, dx + \int_{1}^{3} (-x^2 + 4x - 3) \, dx + \int_{3}^{5} (x^2 - 4x + 3) \, dx \]
\[ = \left( \frac{1}{3}x^3 - 2x^2 + 3x \right) \bigg|_{0}^{1} - \left( \frac{1}{3}x^3 - 2x^2 + 3x \right) \bigg|_{1}^{2} + \left( \frac{1}{3}x^3 - 2x^2 + 3x \right) \bigg|_{3}^{5} \]
\[ = \left( \frac{1}{3} - 2 + 3 \right) - (9 - 18 + 9) + \left( \frac{1}{3} - 2 + 3 \right) + \left( \frac{125}{3} - 50 + 15 \right) - (9 - 18 + 9) \]
\[ = \frac{28}{3} \]
(h) \[ \int_{4}^{9} \frac{16 + t}{t^2} \, dt \]

**SOLUTION:** \[ \int_{4}^{9} \frac{16 + t}{t^2} \, dt = \int_{4}^{9} 16 t^{-2} + t^{-1} \, dt = -16 t^{-1} + \log t \bigg|_{4}^{9} = \frac{20}{9} + \log \frac{9}{4} \]

(2) Solve the differential equation \( \frac{du}{dx} = 8x^3 + 3x^2 - 3 \) with initial condition \( y(1) = 1 \).

**SOLUTION:** Since \( \frac{du}{dx} = 8x^3 + 3x^2 - 3 \), then

\[ y = \int (8x^3 + 3x^2 - 3) \, dx = 2x^4 + x^3 - 3x + C \]

Thus \( y(1) = 0 + C \), and so \( C = 1 \). Therefore, \( y = 2x^4 + x^3 - 3x + 1 \).

(3) Given that \( f''(x) = x^3 - 2x + 1, \) \( f'(0) = 1 \), and \( f(0) = 0 \), find \( f' \) and then find \( f \).

**SOLUTION:** Let \( g(x) = f'(x) \). The statement gives that \( g'(x) = x^3 - 2x + 1, g(0) = 1 \). From this initial value problem, we get \( g(x) = \frac{1}{4}x^4 - x^2 + x + C \). Then \( g(0) = 1 \) gives \( C = 1 \), so \( f'(x) = g(x) = \frac{1}{4}x^4 - x^2 + x + 1 \).

Now we have a new initial value problem to find \( f \), namely \( f'(x) = \frac{1}{4}x^4 - x^2 + x + 1 \) and \( f(0) = 0 \). So we get that \( f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C \). Then \( f(0) = 0 \) gives \( C = 0 \), so

\[ f(x) = \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + x. \]

(4) If \( G(x) = \int_{1}^{x} \tan t \, dt \), find \( G(1) \) and \( G'(\pi/4) \).

**SOLUTION:** By definition, \( G(1) = \int_{1}^{1} \tan t \, dt = 0 \). By FTC II, \( G'(x) = \tan x \), so \( G'(\pi/4) = \tan(\pi/4) = 1 \).

(5) Find a formula for the function represented by the integral: \( \int_{2}^{x} (t^2 - t) \, dt \).

**SOLUTION:** \[ \int_{2}^{x} (t^2 - t) \, dt = \left( \frac{1}{3}t^3 - \frac{1}{2}t^2 \right) \bigg|_{2}^{x} = \frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{2}{3} \]

(6) Express the antiderivative \( F(x) \) of \( f(x) \) as an integral, given that \( f(x) = \sqrt{x^4 + 1} \) and \( F(3) = 0 \).

**SOLUTION:** The antiderivative \( F(x) \) of \( f(x) = \sqrt{x^4 + 1} \) satisfying \( F(3) = 0 \) is

\[ F(x) = \int_{3}^{x} \sqrt{t^4 + 1} \, dt \]

(7) Calculate the derivative: \( \frac{d}{dx} \int_{1}^{x^3} \tan t \, dt \).

**SOLUTION:** By combining FTC II and the chain rule. Let \( G(x) = \int_{1}^{x^3} \tan t \, dt \), \( A(x) = \int_{1}^{x} \tan t \, dt \), and \( g(x) = x^3 \). Then \( G(x) = A(g(x)) \), so we can use the chain rule.

\[ G'(x) = A'(g(x))g'(x) = \tan x^3(3x^2) = 3x^2 \tan x^3 \]