PRELIM 2 REVIEW QUESTIONS
Math 1910 Section 205/209

(1) Calculate the following integrals.

(a) \( \int_0^1 \sqrt{1 - x^2} \, dx \)

SOLUTION: This is just the area under a semicircle of radius 1, so \( \frac{\pi}{2} \).

(b) \( \int \sin^2(x) \cos^4(x) \, dx \)

SOLUTION:

\[
\int \sin^2(x) \cos^4(x) \, dx = \int (1 - \cos^2(x)) \cos^4(x) \, dx \\
= \int \cos^4(x) \, dx - \int \cos^6(x) \, dx
\]

Now use the reduction formula.

\[
\int \cos^4(x) \, dx = \frac{\sin(x) \cos^3(x)}{4} + \frac{3}{4} \int \cos^2(x) \, dx \\
= \frac{\sin(x) \cos^3(x)}{4} + \frac{3}{4} \int \frac{1}{2} (1 + \cos(2x)) \, dx \\
= \frac{\sin(x) \cos^3(x)}{4} + \frac{3}{4} \left( \frac{1}{2} x + \frac{1}{2} \sin(2x) \right) + C
\]

\[
\int \cos^6(x) \, dx = \frac{\sin(x) \cos^5(x)}{6} + \frac{5}{6} \int \cos^4(x) \\
= \frac{\sin(x) \cos^5(x)}{6} + \frac{5}{6} \left( \frac{\sin(x) \cos^3(x)}{4} + \frac{3}{4} \left( \frac{1}{2} x + \frac{1}{2} \sin(2x) \right) + C \right)
\]

Therefore, the answer is

\[
\frac{\sin(x) \cos^3(x)}{4} + \frac{3}{4} \left( \frac{1}{2} x + \frac{1}{2} \sin(2x) \right) - \left( \frac{\sin(x) \cos^5(x)}{6} + \frac{5}{6} \left( \frac{\sin(x) \cos^3(x)}{4} + \frac{3}{4} \left( \frac{1}{2} x + \frac{1}{2} \sin(2x) \right) + C \right)
\]

(c) \( \int \sin^5(x) \cos^4(x) \, dx \)

SOLUTION: This one looks like the reduction formula, but it’s just substitution!

\[
\int \sin^5(x) \cos^4(x) \, dx = \int \sin(x) (1 - \cos^2(x))^2 \cos^4(x) \, dx \\
= \int \sin(x) \cos^4(x) \left( 1 - 2 \cos^2(x) + \cos^4(x) \right) \, dx \\
= \int \sin(x) \cos^4(x) - 2 \sin(x) \cos^6(x) + \sin(x) \cos^8(x) \, dx
\]
Set \( u = \cos(x) \), \( du = -\sin(x) \, dx \). Therefore, we get

\[
\int \sin^5(x) \cos^4(x) \, dx = - \int u^4 - 2u^6 + u^8 \, du
\]

\[
= - \frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C
\]

\[
= \frac{-\cos^5(x)}{5} + \frac{2\cos^7(x)}{7} - \frac{\cos^9(x)}{9} + C
\]

(d) \( \int \tan^6(x) \sec^4(x) \, dx \)

**SOLUTION:**

\[
\int \tan^6(x) \sec^4(x) \, dx = \int \sec^4(x) \tan^6(x)(\tan^2(x) + 1) \, dx
\]

Let \( u = \tan(x) \), \( du = \sec^2(x) \, dx \).

\[
\int \tan^6(x) \sec^4(x) \, dx = \int u^6(u^2 + 1) \, du
\]

\[
= \frac{u^9}{9} + \frac{u^7}{7} + C
\]

\[
= \frac{\tan(x)^9}{9} + \frac{\tan(x)^7}{7} + C
\]

(e) \( \int \cot^5(x) \csc^5(x) \, dx \)

**SOLUTION:**

\[
\int \cot^5(x) \csc^5(x) \, dx = \int \cot(x) \csc(x)(\csc^2(x) - 1)^2 \csc^4(x) \, dx
\]

Let \( u = \csc(x) \), \( du = -\cot(x) \csc(x) \, dx \).

\[
\int \cot^5(x) \csc^5(x) \, dx = - \int (u^2 - 1)^2 u^4 \, du
\]

\[
= - \int (u^4 - 2u^2 + 1) u^4 \, du
\]

\[
= - \int u^8 - 2u^6 + u^4 \, du
\]

\[
= \frac{-\csc^9(x)}{9} + \frac{2\csc^7(x)}{7} - \frac{\csc^5(x)}{5} + C
\]

(f) \( \int \frac{x}{\sqrt{4 - x^2}} \, dx \)
SOLUTION: Let \( x = 2 \sin \theta \), \( dx = 2 \cos \theta \, d\theta \).

\[
\int \frac{x}{\sqrt{4 - x^2}} \, dx = \int \frac{2\sin \theta}{\sqrt{4 - 4\sin^2 \theta}} (2\cos \theta \, d\theta) = \int 2\sin \theta \, d\theta = -2\cos \theta + C = -2\sqrt{4 - x^2} + C
\]

In the last step, we have \( x/2 = \sin \theta = \text{opposite} / \text{hypotenuse}, \text{so} \cos \theta = \sqrt{4 - x^2} \) (draw a triangle).

(g) \( \int \frac{\cosh(x)}{\sinh^2(x)} \, dx \)

SOLUTION: Let \( u = \sinh(x) \) and \( du = \cosh(x) \, dx \). Then

\[
\int \frac{\cosh(x)}{\sinh^2(x)} \, dx = \int \frac{du}{u^2} = -u^{-1} + C = -\frac{1}{\sinh(x)} + C.
\]

(h) \( \int \sin^7(x) \cos^2(x) \, dx \)

SOLUTION: Let \( u = \cos(x) \), \( du = -\sin(x) \, dx \). Then

\[
\int \sin^7 x \cos^2 x \, dx = \int \sin(x)(1 - \cos^2(x))^3 \cos^2(x) \, dx = -\int (1 - u^2)^3u^2 \, du = -\int u^2 - 3u^4 + 3u^6 - u^8 \, du = \frac{u^3}{3} + \frac{3u^5}{5} - \frac{3u^7}{7} + \frac{u^9}{9} + C = -\frac{\cos(x)^3}{3} + \frac{3\cos(x)^5}{5} - \frac{3\cos(x)^7}{7} + \frac{\cos(x)^9}{9} + C
\]

(i) \( \int \frac{3x^2}{\sqrt{x^2 - 1}} \, dx \)

SOLUTION: Let \( x = \sec(\theta) \), \( dx = \tan \theta \sec \theta \, d\theta \). Then

\[
\int \frac{3x^2}{\sqrt{x^2 - 1}} \, dx = \int \frac{3\sec^2(\theta)}{\sqrt{\sec^2 \theta - 1}} \tan \theta \sec \theta \, d\theta = \int 3\sec^3 \theta \, d\theta
\]
Now integrate by parts, with \( u = \sec \theta \), \( du = \sec \theta \tan \theta \, d\theta \) and \( v = \tan \theta \), \( dv = \sec^2 \theta \, d\theta \).

\[
3 \int \sec^3 \theta \, d\theta = 3 \left( \sec \theta \tan \theta - \int \tan^2 \theta \sec \theta \, d\theta \right)
\]

\[
\Rightarrow 3 \sec^3 \theta \, d\theta = 3 \left( \sec \theta \tan \theta - \int (\sec^2 \theta - 1) \sec \theta \, d\theta \right)
\]

\[
\Rightarrow 3 \int \sec^3 (\theta) \, d\theta = 3 \sec \theta \tan \theta + 3 \sec \theta \, d\theta - 3 \int \sec^3 \theta \, d\theta
\]

\[
\Rightarrow 6 \int \sec^3 (\theta) \, d\theta = 3 \sec \theta \tan \theta + 3 \ln |\tan \theta + \sec \theta| + C
\]

\[
\Rightarrow 3 \int \sec^3 \theta \, d\theta = \frac{\sec \theta}{2} + \frac{1}{2} \ln |\tan \theta + \sec \theta| + C.
\]

Therefore, the final answer is

\[
\frac{3x}{2} + \frac{3}{2} \ln \sqrt{x^2 - 1} + x + C.
\]

(j) \( \int \frac{\cosh(x)}{3 \sinh(x) + 4} \, dx \)

SOLUTION: Let \( u = 3 \sinh(x) + 4, \, du = 3 \cosh(x) \, dx \). Then

\[
\int \frac{\cosh(x)}{3 \sinh(x) + 4} \, dx = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |3 \sinh(x) + 4| + C.
\]

(k) \( \int \frac{x^2 + 11x}{(x - 1)(x + 1)^2} \, dx \)

SOLUTION: Perform partial fractions decomposition

\[
\frac{x^2 + 11x}{(x - 1)(x + 1)^2} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{Cx + D}{(x + 1)^2}
\]

Clear denominators first.

\[
x^2 + 11x = A(x + 1)^2 + B(x - 1)(x + 1) + (Cx + D)(x - 1)
\]

Plug in \( x = 1 \) to get \( A = 3 \). Plug in \( x = -1 \) to get \( 5 = D - C \). Plug in \( x = 0 \) to get \( D = B - 3 \). Plug in \( x = 2 \) to get \( -1 = 3B + 2C + D \). Substitute the previous equations to get \( B = 3, \, C = -5, \, \) and \( D = 0 \). So

\[
\int \frac{x^2 + 11x}{(x - 1)(x + 1)^2} \, dx = \int \frac{3}{x - 1} + \frac{3}{x + 1} - \frac{5x}{(x + 1)^2} \, dx
\]

\[
= 3 \ln |x - 1| + 3 \ln |x + 1| - 5 \int \frac{x}{(x + 1)^2} \, dx
\]

\[
= 3 \ln |x - 1| + 3 \ln |x + 1| - 5 \ln |x + 1| - \frac{5}{x + 1} + C
\]

(l) \( \int \frac{3x^2 - 2}{x - 4} \, dx \)
SOLUTION: Do long division. Divide $3x^2 - 2$ by $x - 4$ to get

$$
\int \frac{3x^2 - 2}{x - 4} dx = \int (3x + 12) + \frac{46}{x - 4} dx = \frac{3x^2}{2} + 12x + 46 \ln |x - 4| + C.
$$

(m) $\int \coth^2(1 - 4t) dt$

SOLUTION: Let $u = 1 - 4t$, $du = -4 dt$. Then

$$
\int \coth^2(1 - 4t) dt = -\frac{1}{4} \int \coth^2(u) du = -\frac{1}{4} \int \csch^2(u) + 1 du = -\frac{1}{4} (-\coth(u) + u + C) = \frac{1}{4} \coth(1 - 4t) - (1 - 4t) + C.
$$

(n) $\int \frac{1}{x^2 + 4x - 5} dx$

SOLUTION: Perform partial fractions decomposition:

$$
\frac{1}{x^2 + 4x - 5} = \frac{A}{x + 5} + \frac{B}{x + 1} \Rightarrow 1 = A(x - 1) + B(x + 5)
$$

Set $x = 1 \Rightarrow B = 1/6$. Set $x = -5 \Rightarrow A = -1/6$. Then

$$
\int \frac{1}{x^2 + 4x - 5} dx = -\frac{1}{6} \int \frac{1}{x + 5} - \frac{1}{x - 1} dx = -\frac{1}{6} \left( \ln |x + 5| - \ln |x - 1| + C \right).
$$

(2) Find the volume of the solid obtained by rotating $y = x\sqrt{1 - x^2}$ about the $y$-axis.

SOLUTION: Using the cylindrical shells method:

$$
V = \int_0^1 2\pi x \left( x\sqrt{1 - x^2} \right) dx = 2\pi \int_0^1 x^2 \sqrt{1 - x^2} dx
$$

Let $x = \sin \theta$, $dx = \cos \theta d\theta$. Then

$$
V = 2\pi \int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta
$$

Let $u = \sin \theta$, $du = \cos \theta d\theta$. Then

$$
V = 2\pi \int_0^1 u^2 du = 2\pi \frac{u^3}{3} \bigg|_0^1 = \frac{2}{3} \pi.
$$
(3) Find the arc length of the graph of \( y = \tan(x) \) over the interval \([0, \pi/4]\).

**SOLUTION:** Plug in to the arc length formula.

\[
\int_0^{\pi/4} \sqrt{1 + (y')^2} \, dx = \int_0^{\pi/4} \sqrt{1 + \sec^4(x)} \, dx.
\]

No antiderivative exists, so numerical integration techniques must be used.

(4) Suppose that a random variable \( X \) is distributed with density \( p(x) = C\sqrt{1 - x^2} \) on \([-1, 1]\). Find \( C \) such that \( p(x) \) defines a probability density function, and compute \( P(-1/2 \leq X \leq 1) \).

**SOLUTION:** To find \( C \), set

\[
1 = \int_{-1}^{1} C\sqrt{1 - x^2} \, dx
\]

Then evaluate the integral on the right and solve for \( C \). To evaluate the integral, let \( x = \sin \theta \), \( dx = \cos \theta \, d\theta \). Then

\[
\int_{-1}^{1} C\sqrt{1 - x^2} \, dx = C \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta
\]

\[
= C \int_{-\pi/2}^{\pi/2} \frac{1}{2} (1 + \cos(2\theta)) \, d\theta = C \int_{-\pi/2}^{\pi/2} 1 \, d\theta + C \int_{-\pi/2}^{\pi/2} \cos(2\theta) \, d\theta
\]

\[
= C \frac{\pi}{2} + C \left( \frac{1}{2} \sin(2\theta) \right) \bigg|_{-\pi/2}^{\pi/2}
\]

\[
= C \frac{\pi}{2} + C \left( \sin \pi - \sin(-\pi) \right)
\]

\[
= C \frac{\pi}{2}.
\]

Therefore, \( C = 2/\pi \).

Now to find the probability, we’ve already computed the antiderivative of \( p(x) \), so we just need to substitute new bounds. The probability is

\[
P(-1/2 \leq X \leq 1) = \int_{-1/2}^{1} \frac{2}{\pi} \sqrt{1 - x^2} \, dx = \frac{2}{\pi} \left( \frac{1}{2} \sin^{-1}(x) + \frac{x\sqrt{1 - x^2}}{2} \right) \bigg|_{-1/2}^{1} = \frac{2}{3} + \frac{\sqrt{3}}{4\pi}
\]

(5) Find \( C \) such that \( p(x) = Ce^{-x}e^{-e^{-x}} \) is a probability density function on \((-\infty, \infty)\).

**SOLUTION:** This is one of the homework questions from last week (also it was on the quiz). Go look at the homework solutions on blackboard. The answer is \( C = 1 \).

(6) Suppose that a random variable \( X \) is distributed with density \( p(x) = x^2e^{-x^2} \) on \((-\infty, \infty)\). Find the mean of \( X \).

**SOLUTION:**

\[
\mu = \int_{-\infty}^{\infty} xp(x) \, dx = \int_{-\infty}^{\infty} x^3e^{-x^2} \, dx
\]

This is an odd function integrated over a symmetric domain, so the integral is \( 0 \).
(7) Suppose that a random variable $X$ is distributed with density $p(x) = \frac{1}{r}e^{-x/r}$ on $(0, \infty)$. Find the mean of $X$.

**Solution:** This was on the homework last week, and I also did it in class. Go see the homework solutions on blackboard. The answer is $\mu = r$.

(8) Calculate $T_6$ for the integral $I = \int_0^2 x^3 \, dx$.

**Solution:** $\Delta x = (2 - 0) / 6 = \frac{1}{3}$. This is tedious, but easy to do. The answer is $T_6 = \frac{111}{27}$.

(a) Is $T_6$ too large or too small? Explain graphically.

**Solution:** Between $x = 0$ and $x = 2$, the graph of $y = x^3$ is concave up, so trapezoid rule overestimates the area under the graph; the trapezoids are above the graph.

(b) Show that $K_2 = |f''(2)|$ may be used in the error bound and find a bound for the error.

**Solution:** $K_2$ is the max value of $|f''(x)|$ on the interval $[0, 2]$. $f(x) = x^3$, so $f''(x) = 6x$. Therefore, the maximum value of $|f''(x)|$ on the interval $[0, 2]$ happens at $x = 2$, and $K_2 = |f''(2)| = 12$. Finally,

$$\text{Error} \leq \frac{K_2(b-a)}{6n^2} = \frac{12(2-0)}{6(6^2)} = \frac{24}{216} = \frac{1}{9}.$$

(c) Evaluate $I$ and check that the actual error is less than the bound computed in (b).

**Solution:** This is easy to integrate.

$$\int_0^2 x^3 \, dx = \frac{x^4}{4} \bigg|_0^2 = 4$$

So the actual error is

$$\text{Error} = \left| 4 - \frac{111}{27} \right| = \frac{1}{9}.$$

This is less than the error bound, which says that the error is at most $1/9$.

(9) Radium-226 has a half-life of 1590 years. Consider a mass of 100 mg of Radium-226.

(a) What is the mass of Radium remaining after 1000 years?

**Solution:** The equation for radioactive decay is exponential decay,

$$m(t) = m_0e^{-t/T}$$

where $m_0$ is the initial mass, $m_0 = 100$ mg, $m(t)$ is the number of milligrams remaining after $t$ years, and $T$ is the half life, $T = 1500$ years. Then

$$m(1000) = 100e^{-1000/1590} \approx 187.5.$$
(b) When will the mass of Radium be 10 mg?

**Solution:** We want to know for which \( t \) does \( m(t) = 10 \) mg. So set

\[
10 = 100e^{-t/1590} \implies \frac{1}{10} = \frac{-t}{1590} \implies t = -1590 \ln(1/10).
\]

(10) Show that \( \int_1^\infty e^{-x^2} \, dx \) converges using the Comparison Theorem.

**Solution:** The comparison theorem says that

\[
f(x) \leq g(x) \implies \int_1^\infty f(x) \, dx \leq \int_1^\infty g(x) \, dx.
\]

On the interval \([1, \infty)\), \( e^{-x^2} \leq e^{-x} \). Therefore,

\[
\int_1^\infty e^{-x^2} \, dx \leq \int_1^\infty e^{-x} \, dx = -e^{-x}\bigg|_1^\infty = 1.
\]

Hence, the integral converges.