§9.1 (Arc Length and Surface Area)

1. The arc length of \( f(x) \) on the interval \([a, b]\) is \( \int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx \).

2. The surface area of the surface obtained by rotating the graph of \( f(x) \) around the \( x \)-axis for \( a \leq x \leq b \) is \( 2\pi \int_{a}^{b} f(x) \sqrt{1 + f'(x)^2} \, dx \).

3. The \( n \)-th Taylor Polynomial centered at \( x = a \) for the function \( f \) is

\[
T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.
\]

4. The error for the \( n \)-th Taylor Polynomial is

\[
|T_n(x) - f(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}.
\]

5. Taylor’s Theorem says that

\[
R_n(x) = T_n(x) - f(x) = \frac{1}{n!} \int_{a}^{x} (x-u)^n f^{(n+1)}(u) \, du.
\]

6. A differential equation is like a normal equation, except you solve a differential equation for a function instead of a number.

7. The order of a differential equation is the highest derivative of \( y \) appearing in the equation. What are the orders of the following equations?

<table>
<thead>
<tr>
<th>Equation</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( y' = x^2 )</td>
<td>1</td>
</tr>
<tr>
<td>(b) ( (y')^3 + yy' = \sin x )</td>
<td>1</td>
</tr>
<tr>
<td>(c) ( y'' = y^2 )</td>
<td>2</td>
</tr>
<tr>
<td>(d) ( y''' + x^4y' = 2 )</td>
<td>3</td>
</tr>
</tbody>
</table>

8. The technique for solving a differential equation where you move all the \( x \)-terms to one side and all of the \( y \)-terms to the other side is called Separation of Variables.
(1) For the curve curve \( y = \ln(\cos x) \) over the interval \([0, \pi/4]\), set up an integral to calculate:

(a) the arc length.

**SOLUTION:** First, calculate

\[
1 + (y')^2 = 1 + \tan^2(x) = \sec^2(x),
\]

so the arc length is

\[
\int_0^{\pi/4} \sqrt{1 + (y')^2} \, dx = \int_0^{\pi/4} \sqrt{\sec^2(x)} \, dx = \int_0^{\pi/4} \sec(x) \, dx = \ln |\sec(x) + \tan(x)| \Big|_0^{\pi/4} = \ln |\sqrt{2} + 1|
\]

(b) the surface area when rotated around the \(x\)-axis.

**SOLUTION:** As in the previous part, we have

\[
1 + (y')^2 = \sec^2(x)
\]

Therefore, plug into the arc length formula

\[
\text{Surface Area} = 2\pi \int_0^{\pi/4} y \sqrt{1 + (y')^2} \, dx = 2\pi \int_0^{\pi/4} \ln(\cos(x)) \sec(x) \, dx
\]

(2) Approximate the arc length of the curve \( y = \sin(x) \) over the interval \([0, \pi/2]\) using the midpoint rule \(M_8\).

**SOLUTION:** Since \( y = \sin(x) \), we have

\[
1 + (y')^2 = 1 + \cos^2(x)
\]

Therefore, \( \sqrt{1 + (y')^2} = \sqrt{1 + \cos^2(x)} \), and the arc length over \([0, \pi/2]\) is

\[
\int_0^{\pi/2} \sqrt{1 + \cos^2(x)} \, dx.
\]

Let \( f(x) = \sqrt{1 + \cos^2(x)} \). \(M_8\) is the midpoint approximation with eight subdivisions. So

\[
\Delta x = \frac{\pi/2 - 0}{8} = \frac{\pi}{16}
\]

\[
x_i = 0 + (i - \frac{1}{2})\Delta x \quad \text{for } i = 1, 2, \ldots, 8
\]

\[
y_i = f \left( (i - \frac{1}{2})\Delta x \right)
\]

\[
M_8 = \sum_{i=1}^{8} y_i \Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_8)\Delta x
\]
\begin{array}{ccc}
i & x_i & f(x_i) = y_i \\
1 & 0.5 & 1.41081 \\
2 & 1.5 & 1.3841 \\
3 & 2.5 & 1.3333 \\
4 & 3.5 & 1.26394 \\
5 & 4.5 & 1.18425 \\
6 & 5.5 & 1.10554 \\
7 & 6.5 & 1.04128 \\
8 & 7.5 & 1.00479 \\
\end{array}

The final answer is that the arc length is approximately \(1.9101\).

(3) Find the Taylor polynomials \(T_2(x)\) and \(T_3(x)\) for \(f(x) = \frac{1}{1+x}\) centered at \(a = 1\).

\textbf{SOLUTION:} We need to take a few derivatives, and then plug in \(a = 1\) to each one.

\begin{array}{c|c|c}
n & \text{n-th derivative } f^{(n)}(x) & f^{(n)}(a) \\
\hline
0 & f(x) = \frac{1}{1+x} & f(1) = 1/2 \\
1 & f'(x) = \frac{-1}{(1+x)^2} & f'(1) = -1/4 \\
2 & f''(x) = \frac{2}{(1+x)^3} & f''(1) = 1/4 \\
3 & f'''(x) = \frac{-6}{(1+x)^4} & f'''(1) = -3/8 \\
\end{array}

Then plug these values into the formula for the Taylor polynomial.

\[
T_2(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8}
\]

\[
T_3(x) = \frac{1}{2} - \frac{(x-1)}{4} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{16}
\]

(4) Find \(n\) such that \(|T_n(1.3) - \sqrt{1.3}| \leq 10^{-6}\), where \(T_n\) is the Taylor polynomial for \(\sqrt{x}\) at \(a = 1\).

\textbf{SOLUTION:} By the error formula, we have that

\[
|T_n(1.3) - \sqrt{1.3}| \leq \frac{K_{n+1}(1.3-1)^{n+1}}{(n+1)!}
\]

So we just need to find \(n\) such that

\[
\frac{K_{n+1}(0.3)^{n+1}}{(n+1)!} < 10^{-6},
\]

where \(K_{n+1}\) is the maximum value of the \((n+1)\)-st derivative of \(f(x) = \sqrt{x}\) between 1 and 1.3. Since \(f^{(n+1)}(x)\) is the \((n+1)\)-st derivative of \(\sqrt{x}\), and this always has \(x\) in the
denominator for any $n \geq 0$, this maximum will always occur at $x = 1$. Therefore, in this case,

$$K_{n+1} = |f^{(n+1)}(1)|.$$

So we just need to find $n$ such that

$$\frac{|f^{(n+1)}(1)| \cdot (0.3)^{n+1}}{(n+1)!} < 10^{-6}.$$

The hard part is finding a pattern for the $n$-th derivative of $\sqrt{x}$, but that’s not strictly necessary, although possible. If you keep taking derivatives of $\sqrt{x}$ and plugging into the formula, you find that this is valid for $n \geq 7$.

Alternatively, the general formula for the $n$-th derivative of $\sqrt{x}$ is

$$f^{(n)}(x) = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} x^{\frac{-(2n-1)}{2}}$$

Then you can plug this in to the previous formula.

(5) Find the general solutions of the following differential equations using separation of variables.

(a) $\frac{dy}{dt} - 2y = 1$

**SOLUTION:** First, separate the variables:

$$\frac{dy}{1 + 2y} = dt$$

Then integrate both sides

$$\int \frac{dy}{1 + 2y} = \int dt$$

$$\Rightarrow \frac{1}{2} \ln |1 + 2y| = t + C$$

$$y = -\frac{1}{2} + Ce^{2t}$$

(b) $(1 + x^2)y' = x^3 y$

**SOLUTION:** First, separate the variables:

$$(1 + x^2) \frac{dy}{dx} = x^3 y \Rightarrow \frac{dy}{y} = \frac{x^3 dx}{1 + x^2}$$

Then integrate both sides

$$\int \frac{dy}{y} = \int \frac{x^3 dx}{1 + x^2}$$
Do polynomial long division to the right hand side.

\[
\ln|y| = \int x + \frac{-x}{1 + x^2} \, dx = \frac{x^2}{2} - \frac{\ln|x^2 + 1|}{2} + C
\]

Clear the logarithms, and absorb constants.

\[
y = \frac{Ce^{x^2/2}}{1 + x^2}
\]

(6) Solve the initial value problem \[
\begin{cases}
y' + 2y = 0 \\
y(\ln(2)) = 3
\end{cases}
\]

**SOLUTION:** First, separate variables

\[
\frac{dy}{dx} = -2y \implies \frac{dy}{y} = -2 \, dx.
\]

Then integrate both sides

\[
\int \frac{dy}{y} = \int -2 \, dx \implies \ln|y| = -2x + C.
\]

Now clear the natural logs by exponentiating.

\[
y = Ce^{-2x}
\]

Then plug in the initial value \(y(\ln(2)) = 3\) to get

\[
3 = Ce^{-2\ln(2)} \implies 3 = \frac{C}{4} \implies C = 12.
\]

So the final answer is

\[
y = 12e^{-2x}
\]