§7.1 (Exponential Functions), §7.2 (Inverse functions), §7.3 (Logarithms) September 29, 2016

(1) \( f(x) = b^x \) is increasing if \( b > 1 \) and decreasing if \( b < 1 \).

(2) The derivative of \( f(x) = b^x \) is \( \frac{df}{dx} b^x = b^x \ln(b) \).

(3) \( \frac{d}{dx} e^x = e^x \) and \( \frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)} \) and \( \frac{d}{dx} e^{kx+b} = ke^{kx+b} \).

(4) \( \int e^x \, dx = e^x + C \) and \( \int e^{kx+b} = \frac{1}{k} e^{kx+b} + C \) for constants \( k, b \).

(5) A function \( f \) with domain \( D \) is one to one if \( f(x) = c \) has at most one solution with \( x \in D \).

(6) Let \( f \) have domain \( D \) and range \( R \). The inverse \( f^{-1} \) is the unique function with domain \( R \) and range \( D \) such that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).

(7) The inverse of \( f \) exists if and only if \( f \) is one-to-one on its domain.

(8) Horizontal Line Test: \( f \) is one-to-one if and only if every horizontal line intersects the graph of \( f \) only once.

(9) To find the inverse function, solve \( y = f(x) \) for \( x \) in terms of \( y \).

(10) The graph of \( f^{-1} \) is obtained by reflecting the graph of \( f \) through the line \( y = x \).

(11) If \( f \) is differentiable and one-to-one with inverse \( g \), then for \( x \) such that \( f'(g(x)) \neq 0 \),

\[
g'(x) = \frac{1}{f'(g(x))}.
\]

(12) The inverse of \( f(x) = b^x \) is \( f^{-1}(x) = \log_b(x) \).

(13) Logarithm Rules

- (a) \( \log_b(1) = 0 \) and \( \log_b(b) = 1 \).
- (b) \( \log_b(xy) = \log_b(x) + \log_b(y) \) and \( \log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y) \).
- (c) Change of Base: \( \frac{\log_b(x)}{\log_b(b)} = \log_b(x) \).
- (d) \( \log_b(x^n) = n \log_b(x) \).

(14) \( \frac{d}{dx} \ln(x) = \frac{1}{x} \) and \( \frac{d}{dx} \log_b(x) = \frac{1}{\ln(b) x} \).

(15) \( \int \frac{1}{x} \, dx = \ln|x| + C \).
SOLUTIONS
§7.1 (Exponential Functions), §7.2 (Inverse functions), §7.3 (Logarithms)  September 29, 2016

(1) Calculate the derivative.
   (a) \( f(x) = 7e^{2x} + 3e^{4x} \)
       SOLUTION: \( f'(x) = 14e^{2x} + 12e^{4x} \).
   (b) \( f(x) = e^{x^2} \)
       SOLUTION: \( f'(x) = e^{x^2} \cdot 2x \).
   (c) \( f(x) = 3^x \)
       SOLUTION: \( f'(x) = 3^x \ln(3) \).
   (d) \( f(t) = \frac{1}{1-e^{-3t}} \)
       SOLUTION: \( f'(t) = -3(1-e^{-3t})^{-2}e^{-3t} \).
   (e) \( f(t) = \cos(te^{-2t}) \)
       SOLUTION: \( f'(t) = -\sin(te^{-2t})(e^{-2t} - 2te^{-2t}) \).
   (f) \( \int e^t \sin t \, dt \)
       SOLUTION: Recall that \( \frac{d}{dx} \int_a^x g(t) \, dt = g(f(x))f'(x) \, dx. \) So
       \[ \frac{d}{dx} \int_4^{e^t} \sin t \, dt = \sin(e^t)e^t. \]
   (g) \( f(x) = x \ln x \)
       SOLUTION: \( f'(x) = \ln x + 1 \).
   (h) \( f(x) = \ln(x^5) \)
       SOLUTION: \( f'(x) = \frac{5}{x} \).
   (i) \( f(x) = \ln(\sin(x) + 1) \)
       SOLUTION: \( f'(x) = \frac{\cos(x)}{\sin(x) + 1} \).
(j) \( f(x) = e^{\ln(x)^2} \)

**SOLUTION:** \( f'(x) = e^{(\ln(x)^2)2} \frac{2 \ln(x)}{x} \)

(k) \( f(x) = \log_a(\log_b(x)) \)

**SOLUTION:** \( f'(x) = \frac{\ln(a)}{x \ln(b)} \)

(l) \( f(x) = 16^{\sin x} \)

**SOLUTION:** \( f'(x) = \ln(16) \cos x 16^{\sin x} \).

(2) Calculate the integral.

(a) \( \int (e^x + 2) \, dx \)

**SOLUTION:** \( e^x + 2x + C \)

(b) \( \int \frac{7}{x} \, dx \)

**SOLUTION:** \( 7 \ln |x| + C \)

(c) \( \int e^{4x} \, dx \)

**SOLUTION:** \( \frac{1}{4} e^{4x} + C \)

(d) \( \int \frac{\ln x}{x} \, dx \)

**SOLUTION:** Set \( u = \ln x \), so \( du = \frac{1}{x} \, dx \). Therefore,

\[
\int \frac{\ln x}{x} \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{1}{2} \ln(x)^2 + C.
\]

(e) \( \int \frac{1}{9x - 3} \, dx \)

**SOLUTION:** Let \( u = 9x - 3 \). Then \( du = 9 \, dx \) and substituting gives

\[
\int \frac{1}{9u} \, du = \frac{1}{9} \ln |u| + C = \frac{1}{9} \ln |9x - 3| + C.
\]
(f) \[ \int_2^3 (e^{4t-3}) \, dt \]

**SOLUTION:**
\[ \int_2^3 (e^{4t-3}) \, dt = e^{-3} \int_2^3 e^{4t} \, dt = e^{-3} \left( \frac{1}{4} e^{4t} \right) \bigg|_2^3 = \frac{e^{-3}}{4} \left( e^{12} - e^{8} \right) = \frac{1}{4} (e^9 - e^5) \]

(g) \[ \int e^t \sqrt{e^t + 1} \, dt \]

**SOLUTION:** Let \( u = e^t + 1 \). Then \( du = e^t \, dt \), so the integral becomes
\[ \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (e^t + 1)^{3/2} + C \]

(h) \[ \int e^x \cos e^x \, dx \]

**SOLUTION:** Let \( u = e^x \). Then \( du = e^x \, dx \), so
\[ \int e^x \cos e^x \, dx = \int \cos u = \sin u + C = \sin e^x + C. \]

(i) \[ \int \tan(4x + 1) \, dx \]

**SOLUTION:** First, rewrite the integral as
\[ \int \tan(4x + 1) \, dx = \int \frac{\sin(4x + 1)}{\cos(4x + 1)} \, dx \]
then let \( u = \cos(4x + 1) \), so \( du = -4 \sin(4x + 1) \, dx \). Hence,
\[ \int \frac{\sin(4x + 1)}{\cos(4x + 1)} \, dx = -\frac{1}{4} \int \frac{1}{u} \, du = -\frac{1}{4} \ln |\cos(4x + 1)| + C \]

(j) \[ \int x^3 e^x ^2 \, dx \]

**SOLUTION:** Let \( u = x^2 \). Then \( du = 2x \, dx \), so
\[ \int x^3 e^{x^2} \, dx = \frac{1}{2} \int 3u \, du = \frac{3u}{2 \ln 3} + C = \frac{3x^2}{2 \ln 3} + C. \]
(3) For each function shown below, sketch the graph of the inverse.

\[ y = x \]

\[ y = \sqrt{x^2 + 6x} \text{ for } x \geq 0, \]

\[ y = \frac{1}{\sqrt{x+1}}, b = \frac{1}{4} \]

(4) Calculate \( g(b) \) and \( g'(b) \), where \( g \) is the inverse of \( f \).

(a) \( f(x) = x + \cos x, b = 1 \).

\[ g(1) = 0, g'(1) = 1. \]

(b) \( f(x) = 4x^3 - 2x, b = -2 \).

\[ g(-2) = -1, g'(-2) = \frac{1}{11}. \]

(c) \( f(x) = \sqrt{x^2 + 6x} \text{ for } x \geq 0, b = 4 \).

\[ g(4) = 2, g'(4) = \frac{4}{5}. \]

(d) \( f(x) = \frac{1}{x+1}, b = \frac{1}{4} \).

\[ g(1/4) = 3, g'(1/4) = -16. \]

(5) Which of the following statements are true and which are false? If false, modify the statement to make it correct.

(a) If \( f \) is increasing, then \( f^{-1} \) is increasing.

\[ \text{Solution: True.} \]
(b) If $f$ is concave up, then $f^{-1}$ is concave up.

**SOLUTION:** False. Reflecting the graph of $f$ across the line $y = x$ to get the graph of $f^{-1}$ means that if the graph of $f$ is concave up, then the graph of $f^{-1}$ is concave down.

(c) If $f$ is odd then $f^{-1}$ is odd.

**SOLUTION:** Think of what the graph of an odd function looks like. Reflecting the graph across the line $y = x$ preserves this property.

(d) Linear functions $f(x) = ax + b$ are always one-to-one.

**SOLUTION:** True. The inverse is $f^{-1}(x) = \frac{1}{a}(x - b)$.

(e) $f(x) = \sin(x)$ is one-to-one.

**SOLUTION:** False. The graph of $f(x) = \sin(x)$ fails the horizontal line test. But if we restrict the domain to $(-\pi, \pi)$, then this is true and $\arcsin(x)$ is the inverse of $\sin(x)$ on this domain.