Take-home:

(1) Let $K$ be a finite field extension of $\mathbb{Q}$ and let $G \neq 1$ be a finite abelian group. Prove that there are infinitely many Galois extensions $L$ of $K$, in a fixed algebraic closure of $K$, such that $\text{Gal}(L/K)$ is isomorphic to $G$.

(2) Fix a prime $p$. Let $G$ be the group consisting of functions $f : \mathbb{F}_p \to \mathbb{F}_p$ of the form $f(x) = ax + b$ for some $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$. Find all the irreducible representations of $G$ up to isomorphism (over $\mathbb{C}$).

(3) Fix a prime $p$. Define $G = \text{SL}_2(\mathbb{F}_p)$ and let $H$ be the subgroup consisting of upper triangular matrices. Let $\alpha : \mathbb{F}_p^\times \to \mathbb{C}^\times$ be a homomorphism and let $\chi : H \to \mathbb{C}^\times$ be the homomorphism
\[
\chi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \alpha(a).
\]
Let $\rho$ be the representation of $G$ induced from $\chi$. Show that if $\chi^2 \neq 1$, then $\rho$ is irreducible.

(4) Determine the Galois group of the following polynomials; the last one might be difficult, at least guess what you think the answer is.

(a) $f(x) = x^5 - 5x^2 - 3$.
(b) $f(x) = x^7 - 7x + 3$.

(5) Fix an integer $n \geq 2$ and a positive divisor $d$. Describe the groups
\[
\text{Tor}_{n/m\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z})
\]
for integers $n \geq 0$.

(6) Define $G := \text{Gal}(\mathbb{C}/\mathbb{R})$; it acts on $\mathbb{C}^\times$. Compute $H^2(G, \mathbb{C}^\times)$.

(7) Fix a prime power $q > 1$ and an integer $e \geq 1$. We have defined a norm map $N : \mathbb{F}_q^\times \to \mathbb{F}_q$.

(a) Use Hilbert 90 to prove that the group $\{a \in \mathbb{F}_q^\times : N(a) = 1\}$ has order $(q^n - 1)/(q-1)$. Deduce that $N$ is surjective.

(b) With $G = \text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q)$, prove that $H^n(G, \mathbb{F}_q^\times) = 0$ for all $n \geq 1$. 