Convergence of some time inhomogeneous Markov chains via spectral techniques

Jessica Zuniga
with Laurent Saloff-Coste
Motivation

- Mironov (2002)
  (Not so) random shuffles of RC4.
  - total variation upper bound of $O(n \log n)$ for cyclic-to-random shuffle
  - lower bound of $O(n)$ for cyclic-to-random shuffle

- Mossel, Peres, Sinclair (2004)
  Shuffling by semi-random transpositions.
  - total variation upper bound of $O(n \log n)$ for any semi-random transposition shuffle
  - lower bound of $O(n \log n)$ for cyclic-to-random shuffle

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Basic notation

$V$ a finite set equipped with $(K_n)_{n=1}^{\infty}$ s.t.

$$K_n(x, y) \geq 0 \text{ and } \sum_y K_n(x, y) = 1.$$ 

The associated Markov chain: $X = (X_n)_{n=0}^{\infty}$

$$P(X_n = x | X_{n-1} = y, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0)$$

$$= P(X_n = x | X_{n-1} = y)$$

$$= K_n(x, y)$$

Let $\mu_0$ be the distribution of $X_0$. The distribution $\mu_n$ of $X_n$ is

$$\mu_n(x) = \sum_{y \in V} \mu_0(x)K_{0,n}(x, y)$$

where $K_{n,n} = 1$ and for $m \geq n$. 
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K_{n,m}(x, y) = \sum_{z \in V} K_{n,m-1}(x, z) K_m(z, y).
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\( K_n \) are matrices: \( K_{n,m} = K_{n+1} \cdots K_m \).
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Major assumption: There exists a measure $\pi$ that is invariant for the sequence $(K_n)_{1}^{\infty}$, i.e.

$$\pi K_n = \pi.$$ 

Large class of examples: random walks on groups.

Let $V = G$ a finite group equipped with a probability measure $\rho$. The Markov kernel

$$K(x, y) = \rho(x^{-1}y)$$

has $\pi = 1/|G|$ as an invariant measure.
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**Large class of examples:** random walks on groups.

Let $V = G$ a finite group equipped with a probability measure $p$. The Markov kernel

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Any sequence of probability measures $(p_i)_1^\infty$ on $G$ gives a sequence $(K_i)_1^\infty$ of Markov kernels with invariant measure $\pi$. 

\[ K_n, m(x, y) = p_n \ast \cdots \ast p_m(x^{-1}y) \]
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$$K_{n,m}(x, y) = p_{n+1} \ast \cdots \ast p_{m}(x^{-1}y)$$

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$$u \ast v(x) = \sum_{y \in G} u(y)v(y^{-1}x).$$
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**Definition:** Let $Q = \{Q_1, \ldots, Q_n\}$ have invariant measure $\pi$. $(Q, \pi)$ is ergodic if for any $(K_i)_1^\infty$ with invariant measure $\pi$ and $K_i \in Q$ for infinitely many $i$ then

$$\lim_{n \to \infty} K_{0,n}(x, z) - K_{0,n}(y, z) = 0$$

for all $x, y, z \in V$.

**Remark:**

- Let $Q = \{Q\}$. $(Q, \pi)$ is ergodic is stronger than

$$\forall x, y, z \in V, \lim_{n \to \infty} Q^{(n)}(x, z) - Q^{(n)}(y, z) = 0.$$
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Spectral analysis

\((K, \pi)\) is a linear map on \(\ell^2(V, \pi)\).

Let \(u, v \in \ell^2(V, \pi)\).

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Ku = \sum_{y \in V} K(y, x)u(y)
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- \(\langle u, v \rangle = \sum_{x \in V} u(x)v(x)\pi(x)\)
- \(K^*\) has associated Markov kernel

\[K^*(x, y) = \pi(y)K(y, x)/\pi(x)\]
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Distances on $\ell^2(V, \pi)$:

$$d_{TV}(\mu, \pi) = \sup_{A \in V} |\mu(A) - \pi(A)|$$

$$d_2(\mu, \pi) = \sum_{y \in V} \left|\frac{\mu(y)}{\pi(y)} - 1\right|^2 \pi(y)$$
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Jensen's inequality: $2d_{TV}(\mu, \pi) \leq d_2(\mu, \pi)$.
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- \(K^\ast\) has associated Markov kernel

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Eigenvalues of $K$:

$$1 = \beta_0(K) \geq \beta_1(K) \geq \cdots \geq \beta_{|V|-1}(K) \geq -1.$$ 

Singular values of $K$ on $\ell^2(V, \pi)$:

$$1 = \sigma_0(K) \geq \sigma_1(K) \geq \cdots \geq \sigma_{|V|-1}(K) \geq 0$$

where

$$\sigma_i(K) = \sqrt{\beta_i(KK^*)} = \sqrt{\beta_i(K^*K)}.$$ 

Theorem: Consider $(K, \pi)$. Let $(\psi_i)_{i=0}^{V-1}$ be a basis of $\ell^2(V, \pi)$ such that $KK^*\psi_i = \sigma_i(K)^2\psi_i$ (assume $\psi_0 = 1$). Then

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Main technical result

**Theorem** Let \((K_i)_{1}^{\infty}\) be a sequence in \(V\) with positive invariant measure \(\pi\). For \(j \geq 1\) and \(0 \leq i \leq |V| - 1\) let \(\sigma_i(K_j)\) be the singular values of \(K_j\) on \(\ell^2(V, \pi)\) then

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d_2(K_0, n(x, \cdot), \pi) \leq (\pi(x)^{-1} - 1)^{1/2} \prod_{1}^{n} \sigma_1(K_j)
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Main idea of proof:
For all \(k = 1, \ldots, |V| - 1\)

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\sum_{i \leq k} \sigma_i(K_1 \cdots K_n) \leq \sum_{i \leq k} \sigma_i(K_1) \cdots \sigma_i(K_n)^2.
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The inequality follows from chapter 3 in *Topics in matrix analysis* by Horn and Johnson.
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Remarks:

- Let $Q = \{Q\}$ have invariant measure $\pi$.
  
  1. $\sigma_1(Q) < 1$ implies $Q^n(x, \cdot) \rightarrow \pi$ for all $x$
  
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2. $\sigma_1(K_1 \cdots K_n) < 1$ but $\sigma_1(K_i) = 1$
**Theorem** Let $Q = \{Q_1, \ldots, Q_k\}$ be a family of kernels on $V$ with invariant measure $\pi$.

- $(Q, \pi)$ is ergodic iff $\sigma_1(Q_j) < 1$ for all $1 \leq j \leq k$.

- If $(Q, \pi)$ is ergodic then for any $(K_i)_{i=1}^\infty$ with invariant measure $\pi$ and infinitely many $K_i \in Q$ we have

$$\forall x, \lim_{n \to \infty} K_{0,n}(x, \cdot) - \pi = 0$$
**Theorem** Let \( Q = \{ Q_1, \ldots, Q_k \} \) be a family of kernels on \( V \) with invariant measure \( \pi \).

- \((Q, \pi)\) is ergodic iff \( \sigma_1(Q_j) < 1 \) for all \( 1 \leq j \leq k \).

- If \((Q, \pi)\) is ergodic then for any \((K_i)_1^\infty\) with invariant measure \( \pi \) and infinitely many \( K_i \in Q \) we have

\[
\forall x, \lim_{n \to \infty} K_{0,n}(x, \cdot) - \pi = 0
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Main technical result simplifies when $V = G$, in this case $\pi = 1/|G|$.

- $(\rho_i)_{i=1}^\infty$ a sequence of probability measures on $G$.

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Let \((p_i)_{i=1}^{\infty}\) be a sequence on \(G\). Then

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d_2(p_0, n, \pi) \leq (|G| - 1)^{1/2} \prod_{i=1}^{n} \sigma_1(p_j)
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\[
d_2(p_0, n, \pi) \leq \left( \sum_{i=1}^{|G|-1} \prod_{j=1}^{n} \sigma_i(p_j)^2 \right)^{1/2}.
\]

Example: \((S_j)_{j=1}^{\infty}\) a sequence of generating sets of \(G\) s.t. \(\text{id} \in S_j\). Set

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K_j(x, y) = \begin{cases} 
1/|S_j| & \text{if } x^{-1}y \in S_j \\
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Well known eigenvalue estimates give us

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\sigma(K_j)^2 \leq 1 - \frac{1}{|S_j|^2 d_j^2}
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Semi-random transpositions

Let $G = S_n$ and $\pi = 1/n!$. For $1 \leq i \leq n$ set

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**Theorem:** For $n > 1$, $c > 0$ and $k \geq n(\log n + c)$

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**Definition:** Let \( r = (r_i)_{1}^{\infty} \) with \( r_i \in \{1, \ldots, n\} \). The \( r \)-semi-random transposition shuffle is associated to \( (p_i)_{1}^{\infty} \) where \( p_i = q_{r_i} \).

\[ p_{0,k}^r = p_1 \ast \cdots \ast p_k : \text{distribution after } k \text{ steps} \]

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Remark:

Information can be lost with the inequality

$$\sum_{j=1}^{k} \sigma_j(K_1 \cdots K_n) \leq \sum_{j=1}^{k} \prod_{i=1}^{n} \sigma_j(K_i)^2.$$ 

Let $r = (r_i)_{1}^{\infty}$ be a sequence of i.i.d. uniform random variables taking values in $\{1, \ldots, n\}$.

$\rho_{0,k}$: random transposition shuffle
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\( \rho_{0,k}^\epsilon \) \: random transposition shuffle

Diaconis-Shahshahani:

For \( c > 0 \), \( k \geq (n/2)(\log n + c) \)

\[ \beta(\rho_{0,k}^\epsilon, \pi) \leq 2^{-\epsilon}. \]
Remark:

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\[ \sum_{j=1}^{k} \sigma_j(K_1 \cdots K_n) \leq \sum_{j=1}^{k} \prod_{i=1}^{n} \sigma_j(K_i)^2. \]

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Semi-random insertions

c_{i,j}: pick card in position \(i\) and insert it into position \(j\).

\[ c_{i,j} = \begin{cases} 
\text{id} & \text{if } i = j \\
(j, j - 1, \ldots, i + 1, i) & \text{if } 1 \leq i < j \leq n \\
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Random insertion measure on \(S_n\):

\[ \tilde{q}(x) = \begin{cases} 
1/n & \text{if } x = \text{id}, \\
2/n^2 & \text{if } x = c_{i,j} \text{ if } |i - j| = 1 \\
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Theorem (Diaconis and Saloff-Coste): For all \(n \geq 28\), \(c > 2\) and all \(k \geq 2n(\log n + c)\)

\[ d_2(\tilde{\pi}(k), \pi) \leq 2e^{-\frac{k}{2}}. \]
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The adjoint \( \tilde{q}_i^*(x) = \tilde{q}_i(x^{-1}) \) inserts uniformly chosen card into position \( i \).
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The adjoint \(\tilde{q}_i^*(x) = \tilde{q}_i(x^{-1})\) inserts uniformly chosen card into position \(i\).

Note that:

- For any \(1 \leq i \leq n\), \(\tilde{q}_i^* + \tilde{q}_i = \tilde{1}\).
Semi-random insertions

For \( i \in \{1, \ldots, n\} \) set

\[
\tilde{q}_i(x) = \begin{cases} 
1/n & \text{if } x = c_{i,j} \text{ for some } j, 1 \leq j \leq n, \\
0 & \text{otherwise.}
\end{cases}
\]

The adjoint \( \tilde{q}_i^*(x) = \tilde{q}_i(x^{-1}) \) inserts uniformly chosen card into position \( i \).

Note that:

- For any \( 1 \leq i \leq n \), \( \tilde{q}_i^* \ast \tilde{q}_i = \tilde{q} \).
- \( \sigma_j(\tilde{q}_i) = \sigma_j(\tilde{q})^{1/2} \), \( 0 \leq j \leq n! - 1 \).
Semi-random insertions

For $i \in \{1, \ldots, n\}$ set

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Note that:

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Definition Let $r = (n)^\infty$ with $r_n \in \{1, \ldots, n\}$. The $r$-semi-random insertion shuffle is associated to $(\tilde{p}_i)^\infty$ where $p_i = \tilde{q}_i$.

$\tilde{p}_{i,k} = \tilde{p}_i \ast \cdots \ast \tilde{p}_k$ - distribution after $k$ steps.
Semi-random insertions

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Note that:

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Definition Let $r = (r_i)_1^\infty$ with $r_i \in \{1, \ldots, n\}$. The $r$-semi-random insertion shuffle is associated to $(\tilde{p}_i)_1^\infty$ where $\tilde{p}_i = \tilde{q}_{r_i}$.

$\tilde{p}_{0,k}^r = \tilde{p}_1 \ast \cdots \ast \tilde{p}_k$ : distribution after $k$ steps.
Semi-random insertions

For $i \in \{1, \ldots, n\}$ set

$$
\tilde{q}_i(x) = \begin{cases} 
1/n & \text{if } x = c_{i,j} \text{ for some } j, 1 \leq j \leq n, \\
0 & \text{otherwise.}
\end{cases}
$$

The adjoint $\tilde{q}_i^*(x) = \tilde{q}_i(x^{-1})$ inserts uniformly chosen card into position $i$.

Note that:

- For any $1 \leq i \leq n$, $\tilde{q}_i^* \ast \tilde{q}_i = \tilde{q}$.
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**Definition** Let $r = (r_i)_1^\infty$ with $r_i \in \{1, \ldots, n\}$. The $r$-semi-random insertion shuffle is associated to $(\tilde{p}_i)_1^\infty$ where $\tilde{p}_i = \tilde{q}_{r_i}$.

$$
\tilde{p}_{0,k} = \tilde{p}_1 \ast \cdots \ast \tilde{p}_k : \text{distribution after } k \text{ steps.}
$$
Semi-random insertions

**Theorem** For any $r = (r_i)_1^\infty, r_i \in \{1, \ldots, n\}$

$$d_2(\tilde{p}_0, 2^k, \pi) \leq d_2(\tilde{q}^{(k)}, \pi).$$

For $n > 28$, $c > 2$ and $k \geq 4n(\log n + c)$

$$d_2(\tilde{p}_0, k, \pi) \leq 2e^{-(c-2)}.$$

**Proof**

$$d_2(\tilde{p}_0, 2^k, \pi)^2 \leq \sum_{m=1}^{2^k} \prod_{i=1}^{m} \sigma_m(q_i)^2.$$
Theorem For any $r = (r_i)_1^\infty$, $r_i \in \{1, \ldots, n\}$

\[
d_2(\tilde{p}_0^{r}, \pi) \leq d_2(\tilde{q}^{(k)}, \pi).
\]

For $n > 28$, $c > 2$ and $k \geq 4n(\log n + c)$

\[
d_2(\tilde{p}_0^{r}, \pi) \leq 2e^{-(c-2)}.
\]

Proof

\[
d_2(\tilde{p}_0^{r}, \pi)^2 \leq \sum_{m=1}^{n!-1} \prod_{i=1}^{2k} \sigma_m(\tilde{q}_{r_i})^2
\]

\[
= \sum_{m=1}^{n!-1} \sigma_m(\tilde{q})^{2k}
\]
**Theorem** For any \( r = (r_i)_{1}^{\infty}, r_i \in \{1, \ldots, n\} \)

\[
d_2(\tilde{p}_{0,2k}^r, \pi) \leq d_2(\tilde{q}^{(k)}, \pi).
\]

For \( n > 28, \ c > 2 \) and \( k \geq 4n(\log n + c) \)

\[
d_2(\tilde{p}_{0,k}^r, \pi) \leq 2e^{-(c-2)}.
\]

**Proof**

\[
d_2(\tilde{p}_{0,2k}^r, \pi)^2 \leq \sum_{m=1}^{n!-1} \prod_{i=1}^{2k} \sigma_m(\tilde{q}_{r_i})^2
\]

\[
= \sum_{m=1}^{n!-1} \sigma_m(\tilde{q})^{2k}
\]

\[
= d_2(\tilde{q}^{(k)}, \pi)
\]
Semi-random insertions

**Theorem** For any $r = (r_i)_{1}^{\infty}$, $r_i \in \{1, \ldots, n\}$

$$d_2(\tilde{p}^r_{0, 2k}, \pi) \leq d_2(\tilde{q}^{(k)}, \pi).$$

For $n > 28$, $c > 2$ and $k \geq 4n(\log n + c)$

$$d_2(\tilde{p}^r_{0, k}, \pi) \leq 2e^{-(c-2)}.$$

**Proof**

$$d_2(\tilde{p}^r_{0, 2k}, \pi)^2 \leq \sum_{m=1}^{n!-1} \prod_{i=1}^{2k} \sigma_m(\tilde{q}_{r_i})^2$$

$$= \sum_{m=1}^{n!-1} \sigma_m(\tilde{q})^{2k}$$

$$= d_2(\tilde{q}^{(k)}, \pi).$$
Semi-random insertions

**Theorem** For any \( r = (r_i)_{1}^{\infty}, r_i \in \{1, \ldots, n\} \)

\[
d_2(\tilde{p}_{0,2k}^r, \pi) \leq d_2(\tilde{q}^{(k)}, \pi).
\]

For \( n > 28, \ c > 2 \) and \( k \geq 4n(\log n + c) \)

\[
d_2(\tilde{p}_{0,k}^r, \pi) \leq 2e^{-(c-2)}.
\]

**Proof**

\[
d_2(\tilde{p}_{0,2k}^r, \pi)^2 \leq \sum_{m=1}^{n!-1} \prod_{i=1}^{2k} \sigma_m(\tilde{q}_{r_i})^2
\]

\[
= \sum_{m=1}^{n!-1} \sigma_m(\tilde{q})^{2k}
\]

\[
= d_2(\tilde{q}^{(k)}, \pi)
\]