Constructing pseudo-Anosov homeomorphisms

Hyungryul Baik, Ahmad Rafiqi, Chenxi Wu

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The question we’re trying to address is: Given a bi-Perron number $\lambda$, can we construct a compact surface $S_g$, and a pseudo-Anosov homeomorphism $\psi : S_g \to S_g$, whose dilatation factor is $\lambda$?
\(\lambda\)-expander: a piecewise-linear continuous map from \(I\) to itself, such that the slope of each piece is either \(\lambda\) or \(-\lambda\).

Post-critically finite: the critical points all have finite orbit.

Thurston proved the following Theorem in his paper ”Entropy in Dimension 1”:

**Theorem (Thurston)**

*Given any weak Perron number \(\lambda\), there is a post-critically finite \(\lambda\)-expander from the unit interval to itself.*
Furthermore, he "thickened" the tent map with slope as the golden ratio, and the tent map with slope as the leading root of
\[ \lambda^4 - \lambda^3 - \lambda^2 - \lambda + 1 = 0 \]
into a two dimensional piecewise-linear map, and glue their \( \omega \)-limit set into closed surfaces. Carvalho-Hall also gave a similar construction for PCF tent maps.
Our strategy is to add further conditions on a PCF $\lambda$-expander that are sufficient to guarantee that its thickening is a pseudo-Anosov dilatation with $\lambda$ as the eigenvalue. We will first describe a general construction, possibly yielding a surface of infinite type, and then prove that our conditions are sufficient to ensure the surface constructed is of finite type.
General construction

\[ x_6 = 1 \]

\[ x_{\phi_3} = x_5 \]

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General construction

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Then we glue the boundary so that $f_0$ and its inverse are both continuous.
The conditions

A $\lambda$-expander $h$ can be thickened into a pseudo-Anosov map on closed surface when it:
1. Permutes the post-critical set.
2. Has an aperiodic incidence matrix.
3. Satisfies the ”one sided condition”: for any $x$ in the post-critical set of $h$, $x = \sup h^{-1}(h(x))$ or $x = \inf h^{-1}(h(x))$. 

The conditions

4. Satisfies the ”alignment condition”: there is a number \( \epsilon \in \{-1, 1\} \), and a function \( \alpha \) from the post-critical set \( X \) to \( \{-1, 1\} \), such that:

(a) If \( h^{-1}(h(x)) \) has more than one point, \( \alpha(x) = -1 \) if \( x = \inf h^{-1}(h(x)) \), \( \alpha(x) = 1 \) if \( x = \sup h^{-1}(h(x)) \).

(b) For critical \( x \in X \), \( \alpha(i) = \begin{cases} -\epsilon & \text{if } x \text{ is a local max.} \\ +\epsilon & \text{if } x \text{ is a local min.} \end{cases} \)

(c) For noncritical \( x \in X \), \( \alpha(i) = \begin{cases} +\epsilon \alpha(h(x)) & \text{if } h'(x) > 0. \\ -\epsilon \alpha(h(x)) & \text{if } h'(x) < 0. \end{cases} \)

All 4 conditions are very easy to check computationally.