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1. Infinite translation surface
2. Linear approaches and rotational components
3. Examples
Infinite translation surface

Translation surface: surface with a chart whose coordinate transforms are all translations.
Infinite translation surface: Translation surface whose metric completion is not a closed surface w/ an abelian differential.
Application:
1. Infinite IET
2. Piecewise linear maps on union of rectangles
Joshua Bowman and Ferrán Valdez introduced the following concepts in their 2013 paper "Wild singularities of flat surfaces" to characterize the topology of isolated singularities of infinite translation surfaces.
Definition (Linear approaches)

Let $(X, \omega)$ be a translation surface and $\epsilon > 0$. On the space

$$\mathcal{L}^\epsilon(X) = \{ \gamma : (0, \epsilon) \to X : \gamma \text{ is a geodesic curve} \}$$

we define the following equivalence relation: $\gamma_1 \in \mathcal{L}^\epsilon(X)$ and $\gamma_2 \in \mathcal{L}^{\epsilon'}(X)$ are called equivalent if $\gamma_1(t) = \gamma_2(t)$ for all $t \in (0, \min\{\epsilon, \epsilon'\})$.

The space

$$\mathcal{L}(X) = \bigsqcup_{\epsilon > 0} \mathcal{L}^\epsilon(X) / \sim$$

is called space of linear approaches of $X$ and the equivalence class $[\gamma]$ of $\gamma \in \mathcal{L}^\epsilon(X)$ is called linear approach.

We give $\mathcal{L}^\epsilon$ the uniform topology and $\mathcal{L}$ the direct limit topology.
An angular sector is a triple \((I, c, i_c)\) such that \(I\) is a non-empty interval, \(c \in \mathbb{R}\) and \(i_c\) is an isometry from \(\{x + iy | x < c, y \in I\} \subseteq \mathbb{C}\) with metric defined by \(e^z dz\) to \(X\).

Define an equivalence relation \(\sim\) as follows: two linear approaches \([\gamma_1]\) and \([\gamma_2]\) are related by \(\sim\) if there is an angular sector \((I, c, i_c)\) such that \(i_c^{-1}\) of the images of \(\gamma_1\) and \(\gamma_2\) are two rays parallel to the real axis.

**Definition (Rotational components)**

An equivalence class under \(\sim\) is called a *rotational component*. We define the *space of rotational components* as \(\tilde{\mathcal{L}}(x) = \mathcal{L}(x)/\sim\). We give \(\tilde{\mathcal{L}}\) the quotient topology.
Chamanara surface: segments with the same letters are glued.

Reza Chamanara, "Affine automorphism groups of surfaces of infinite type", 2014
Example 2

Geometric series decoration
Example 3

Star decoration

Singularity of infinite translation surface
Example 4

Shrinking star decoration

Singularity of infinite translation surface
Singularity of infinite translation surface
Example 5

Singularity of infinite translation surface