(1) Show that if $E[|X|^n] < \infty$, then the characteristic function $\varphi$ of $X$ has a continuous derivative of order $n$ given by
\[
\varphi^{(n)}(t) = \int (ix)^n e^{itx} d\mu(x).
\]

(2) Suppose that $X$ is absolutely continuous with density $f(x) = \frac{c}{x^2 \log |x|} 1\{|x| > 4\}$ for an appropriate normalizing constant $c$.
   
   (a) Show that $E|X| = \infty$.
   
   (b) Show that $X$ has characteristic function $\varphi$ satisfying $\varphi'(0) = 0$.
      (Hint: Express $1 - \varphi(t)$ as an integral over $|x| > 4$ and use the change of variables $y = tx$ to prove that $\frac{1-\varphi(t)}{t} \to 0$ as $t \to 0$. The inequality $|1 - \cos(y)| \leq y^2$ may be useful.)

(3) Let $X_1, X_2, \ldots$ be i.i.d. and let $S_n = X_1 + \ldots + X_n$. Assume that $\frac{1}{\sqrt{n}} S_n$ has a distributional limit as $n \to \infty$. Prove that $E[X_1^2] < \infty$.
   
   (Sketch: Assume $E[X_1^2] = \infty$. Let $X_1', X_2', \ldots$ be an independent copy of the original sequence. Set $Y_i = X_i - X_i', U_i = Y_i 1\{|Y_i| \leq A\}$, $V_i = Y_i 1\{|Y_i| > A\}$, and argue that
\[
P\left(\sum_{i=1}^n Y_i \geq K\sqrt{n}\right) \geq P\left(\sum_{i=1}^n U_i \geq K\sqrt{n}, \sum_{i=1}^n V_i \geq 0\right)
\]
\[
\geq \frac{1}{2} P\left(\sum_{i=1}^n U_i \geq K\sqrt{n}\right) \geq \frac{1}{5}
\]
for any $K > 0$ provided that $A$ and $n$ are sufficiently large.)

(4) Let $X_1, X_2, \ldots$ be i.i.d. with $E[X_i] = 0$ and $E[X_i^2] = \sigma^2 \in (0, \infty)$. Show that
\[
\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}} \Rightarrow Z \sim N(0,1).
\]
(5) Let $X_1, X_2, \ldots$ be independent random variables such that $X_n$ is uniformly distributed on $[-n,n]$. Let $S_n = X_1 + \ldots + X_n$. Show that there is an $\alpha > 0$ such that $n^{-\alpha}S_n$ converges weakly to a normal random variable $V \sim N(\mu, \sigma^2)$ as $n \to \infty$. Identify $\alpha$, $\mu$, and $\sigma^2$.

(6) Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $E[X_i] = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2$. Let $S_n = \sum_{i=1}^n X_i$, $s_n = \text{Var}(S_n)^{\frac{1}{2}}$. Show that if
\[
\lim_{n \to \infty} s_n^{-(2+\delta)} \sum_{i=1}^n E[|X_i - \mu_i|^{2+\delta}] = 0
\]
for some $\delta > 0$, then
\[
\frac{S_n - E[S_n]}{s_n} \Rightarrow Z \sim N(0,1).
\]

(7) Suppose that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are sequences of real numbers with $a_n \to \infty$ and $b_n \to b$ as $n \to \infty$, and that $X_1, X_2, \ldots$ is a sequence of random variables with $a_n(X_n - b_n) \Rightarrow X$. Show that if $g : \mathbb{R} \to \mathbb{R}$ has at continuous derivative at $b$, then
\[
a_n (g(X_n) - g(b_n)) \Rightarrow g'(b)X.
\]