1. Suppose that $X$ has continuous density $f_X$, $P(a \leq X \leq b) = 1$, and $g$ is a strictly monotone function which is differentiable on $(a, b)$. Prove that $Y = g(X)$ has density $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$ for $y$ between $g(a)$ and $g(b)$ and $f_Y(y) = 0$ otherwise. (Hint: Write $F_Y$ in terms of $F_X$, casing out according to whether $g$ is increasing or decreasing.)

2. Suppose that $X$, $Y$, and $Z$ are random variables with $X \overset{d}{=} Y$. Can we conclude that $XZ \overset{d}{=} YZ$? Prove or give a counterexample.

3. Suppose that $X$ is a map from $(\Omega, \mathcal{F})$ to $(S, \mathcal{G})$. Show that if $\mathcal{A}$ generates $\mathcal{G}$, then $X^{-1}(A) := \{X \in A \in \mathcal{A}\}$ generates $\sigma(X)$.

4. Recall that a function $\varphi : \Omega \to \mathbb{R}$ is said to be simple if

$$\varphi(\omega) = \sum_{k=1}^{n} c_k 1_{A_k}(\omega)$$

for some $n \in \mathbb{N}$, $c_1, ..., c_n \in \mathbb{R}$, and $A_1, ..., A_n \in \mathcal{F}$.

a) Show that the class of $\mathcal{F}$-measurable functions is the smallest class containing the simple functions and closed under pointwise limits.

b) Use part a) to conclude that $Y$ is measurable with respect to $\sigma(X)$ if and only if $Y = f(X)$ for some measurable $f : \mathbb{R} \to \mathbb{R}$.

(This result is known as the Doob-Dynkin lemma.)

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex.

a) Show that

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}$$

for every $x < y < z$.

b) Use part a) to deduce that for any $c \in \mathbb{R}$, there exist one-sided derivatives $f'_l(c) \leq f'_r(c)$ defined by

$$\frac{f(c) - f(c - h)}{h} \nearrow f'_l(c), \quad \frac{f(c + h) - f(c)}{h} \searrow f'_r(c) \text{ as } h \downarrow 0.$$  

Use part b) to show that for any $c \in \mathbb{R}$, there is a linear function $l(x)$ which satisfies $l(c) = f(c)$ and $l(x) \leq f(x)$ for all $x \in \mathbb{R}$. 
