Homework 10

(1) Given a probability space \((\Omega, \mathcal{F}, P)\) and events \(A, B \in \mathcal{F}\) with \(P(B) > 0\), we define the conditional probability of \(A\) given \(B\) by \(P(A|B) = \frac{P(A \cap B)}{P(B)}\).

If \(T \sim \text{Exp}(\lambda)\), then it is immediate that for every \(s, t > 0\)

\[(*) \quad P(T > t + s | T > t) = P(T > s).
\]

Show that if \(T\) is a random variable with \(P(T > 0) = 1\) such that \((*)\) holds for all \(s, t > 0\), then \(T \sim \text{Exp}(\lambda)\) for some \(\lambda > 0\).

(2) Let \(N\) have a Poisson distribution with mean \(\lambda\) and let \(X_1, X_2, \ldots\) be an i.i.d. sequence, independent of \(N\), with \(P(X_i = j) = p_j\) where \(p_0, p_1, ..., p_k\) are positive and sum to 1. Let \(N_j = |\{m \leq N : X_m = j\}|\).

Show that \(N_0, N_1, ..., N_k\) are independent and \(N_i \sim \text{Poisson}(\lambda p_i)\).

(3) For \(\lambda > 0\), define the operator \(A\) by

\[
(Af)(k) = \lambda f(k+1) - kf(k).
\]

Show that a random variable \(W\) taking values in \(\mathbb{N}_0\) satisfies \(E[(Af)(W)] = 0\) for all bounded functions \(f\) if and only if \(W \sim \text{Poisson}(\lambda)\).

(4) Let \(A \subseteq \mathbb{N}_0\) and let \(\mathcal{P}_\lambda\) denote probability with respect to the Poisson distribution with mean \(\lambda\).

Show that the unique solution of

\[
\lambda f(k+1) - kf(k) = 1 \{k \in A\} - \mathcal{P}_\lambda(A), \quad f(0) = 0
\]

is given by

\[
f_A(k) = \lambda^{-k} e^{\lambda(k-1)} [\mathcal{P}_\lambda(A \cap U_k) - \mathcal{P}_\lambda(A)\mathcal{P}_\lambda(U_k)]
\]

where \(U_k = \{0, 1, ..., k-1\}, U(0) = \emptyset\).