1. Let $H$ and $K$ be subgroups of a group $G$. Recall that
\[ HK := \{hk \mid h \in H, \ k \in K \}. \]
(a) Give an example in which $H \cup K$ is not a subgroup of $G$.
(b) Give an example in which $HK$ is not a subgroup of $G$.
(c) Show that the following statements are equivalent.
   i. $HK = \langle H \cup K \rangle$.
   ii. $HK$ is a subgroup of $G$.
   iii. $HK = KH$.
(d) Show that if $H \subseteq N_G(K)$ then $HK = KH$.
(e) Show that the converse to (1d) does not hold.
(f) Suppose $H$ and $K$ are both normal in $G$ and $H \cap K = \{1\}$. Show that $hk = kh$ for all $h \in H$, $k \in K$.

2. Let $(I, \leq)$ be a partially order set such that for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Let $\{G_i\}_{i \in I}$ be a family of subgroups of a group $G$ such that if $i \leq j$ then $G_i \subseteq G_j$.

(a) Show that in this case $\bigcup_{i \in I} G_i \leq G$.

(b) Let $\mu_n$ be the group of $n$-th roots of unity in $\mathbb{C}$. Show that
\[ \mu_\infty := \bigcup_{n \geq 1} \mu_n \]

is a subgroup of $S^1$.

3. (a) Let $A, B, C$ be subgroups of a group $D$ such that $B, C \subseteq D$ and $A = B \cap C$. Use the isomorphism theorems to show that
\[ \frac{D/B}{C/A} \cong \frac{D/C}{B/A}. \]
First explain why each quotient makes sense.

(b) Let $\mathbb{F}$ be a field and $n$ a positive integer. If necessary, look up the definition of the projective general and projective special linear groups
\[ \text{PGL}(n, \mathbb{F}) \quad \text{and} \quad \text{PSL}(n, \mathbb{F}). \]

Let
\[ (\mathbb{F}^\times)^{(n)} := \{x \in \mathbb{F}^\times \mid \text{there is } y \in \mathbb{F}^\times \text{ such that } x = y^n\} \]
be the groups of $n$-th powers in $\mathbb{F}^\times$. Deduce that
\[ \text{PGL}(n, \mathbb{F})/\text{PSL}(n, \mathbb{F}) \cong \mathbb{F}^\times/(\mathbb{F}^\times)^{(n)}. \]
4. (a) Recall that if $H$ normalizes $N$, then

$$H/H \cap N \rightarrow HN/N, \quad h(H \cap N) \mapsto hN$$

is an isomorphism. Describe the inverse isomorphism explicitly.

(b) Describe the isomorphism in the Butterfly Lemma explicitly.

5. Let $G$ be a group whose only subgroups are $\{1\}$ and $G$. Show that $G$ is either trivial or cyclic of prime order.

6. Let $G$ be a finite abelian group and $p$ a prime divisor of $|G|$. Prove that $G$ contains an element of order $p$, without appealing to the structure theorem for finite abelian groups. Hint: choose a proper nontrivial subgroup (when possible) and argue by induction.

This exercise proves a result we will use in class when discussing $p$-groups. Cauchy’s Theorem says that the result holds for all finite groups, not just the abelian ones. In class we will deduce Cauchy’s Theorem from the Sylow Theorems.

7. Let $x, y$ and $z$ be integers such that $x$ divides $z$. Prove that

$$\text{lcm}(x, \gcd(y, z)) = \gcd(\text{lcm}(x, y), z).$$

8. Consider the dihedral group of order $2n$.

(a) When $n$ is even, find two subnormal series of length 2 for which $\mathbb{Z}/2$ is one of the slices, but it appears first in one series and last in the other.

(b) Are there such series when $n$ is odd?

9. Let $G$ be a group with a composition series and $H \trianglelefteq G$.

(a) Show that $G$ has a composition series in which $H$ is one of the terms. Deduce that $H$ and $G/H$ have composition series.

(b) The length of some (every) composition series of $G$ is denoted $\ell(G)$. Show that $\ell(G) = \ell(H) + \ell(G/H)$.

(c) If $K$ is another normal subgroup of $G$, show that

$$\ell(HK) = \ell(H) + \ell(K) - \ell(H \cap K).$$

10. (a) Let $\rho = (a_1, \ldots, a_r)$ be an $r$-cycle and $\sigma$ a permutation in $S_n$. Show that

$$\sigma \rho \sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_r)).$$

(b) Describe the conjugacy classes in $S_n$.

11. (a) Compute the conjugacy classes in $A_5$.

(b) Prove that $A_5$ is simple.

12. Let $\Omega$ be a set. An $\Omega$-group is a group $G$ together with a map

$$\Omega \times G \rightarrow G, \quad (\omega, g) \mapsto \omega g$$

such that

$$\omega(gh) = \omega(g) \omega(h)$$

for all $\omega \in \Omega$, $g, h \in G$. Note that this is equivalent to a map $\Omega \rightarrow \text{End}(G)$, where $\text{End}(G)$ denotes the set of all homomorphisms $G \rightarrow G$. Thus, $G$ is a group with a collection of endomorphisms indexed by $\Omega$. An $\Omega$-group is also called a group with operators.
(a) Define suitable notions of Ω-subgroup and homomorphism of Ω-groups.

(b) An Ω-subgroup of an Ω-group $G$ is normal if it is normal as a subgroup of $G$. Let $N$ be such a subgroup. Show that $G/N$ is an Ω-group in such a way that the canonical projection $G \rightarrow G/N$ is a homomorphism of Ω-groups.

(c) Briefly review the isomorphism laws and note that they hold in the context of Ω-groups.

(d) An Ω-group $G$ is simple if it is nontrivial and the only normal Ω-subgroups are $\{1\}$ and $G$. An Ω-composition series of $G$ is a subnormal series whose slices are simple Ω-groups. Review the Butterfly Lemma, Schreier’s Refinement Theorem, and the Jordan-Hölder Theorem, and note that they hold in the context of Ω-groups.

13. Let $G$ be an Ω-group. Prove the following statements, or give a counterexample.

(a) The commutator subgroup $[G, G]$ is an Ω-subgroup.

(b) The center $Z(G)$ is an Ω-subgroup.

14. (a) Give an example of two nonisomorphic groups with isomorphic composition factors.

(b) Let $S_1, \ldots, S_n$ be a collection of simple groups (not necessarily nonisomorphic). Construct a group with those groups for composition factors.