Math 481
Test 1 Solutions
1:25-2:15
February 24, 2005

All proofs should consist of complete sentences. Sentences can be equations. Calculations without words get a zero. You are supposed to learn to write proofs in full.

1) For sets $A, B$ define $A - B = \{x \mid x \in A \text{ and } x \notin B\}$ and $A \triangle B = (A - B) \cup (B - A)$
Prove that for all sets $A, B, C$,
$$A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C).$$

Proof. $x \in A \cap (B \triangle C)$ iff $x \in A$ and $x \in B \triangle C$ iff $x \in A$ and ($x \in B$ and $x \notin C$) or ($x \in C$ and $x \notin B$) iff
$x \in A$ and $x \notin C$ or $x \in B$ and $x \notin C$ or ($x \in A$ and $x \in C$ and $x \notin B$) iff $x \in (A \cap B) \triangle (A \cap C)$

Other style proof. The proof divides into two parts.
Part I is to show that $A \cap (B \triangle C) \subseteq (A \cap B) \triangle (A \cap C)$.
Part 2 is to show that $(A \cap B) \triangle (A \cap C) \subseteq A \cap (B \triangle C)$
Part 1. Suppose that $x \in A \cap (B \triangle C)$. Then $x \in A$ and $x \in (B \triangle C) = (B - C) \cup (C - B)$.
Case 1. $x \in (B - C)$. Then $x \in A \cap B$ and $x \notin A \cap C$. So $x \in (A \cap B) - (A \cap C) \subseteq (A \cap B) \triangle (A \cap C)$
Case 2. $x \in (C - B)$. Then $x \in A \cap C$ and $x \notin A \cap B$. So $x \in (A \cap C) - (A \cap B) \subseteq (A \cap B) \triangle (A \cap C)$

Part 2. Suppose that $x \in (A \cap B) \triangle (A \cap C) = ((A \cap B) - (A \cap C)) \cup (A \cap C - A \cap B)$.
Case 1. $x \in (A \cap B) - (A \cap C)$. Then $x \in A$ and $x \in B$ and $x \notin C$. So $x \in A$ and $x \notin C$ or $x \in B$ and $x \notin C$. So $x \in A \cap (B \triangle C)$
Case 2. $x \in (A \cap C) - (A \cap B)$. Then $x \in A$ and $x \in C$ and $x \notin B$. So $x \in A$ and $x \notin B$. So $x \in A \cap (B \triangle C)$

.2a) State the axiom of unordered pairs, the definition of ordered pair, the axiom of power set, the axiom of union, and the subset construction axiom.

Solution: Look at text
2b) If $A, B$ are sets, the Cartesian product $A \times B$ is the collection of all ordered pairs $(a, b)$ with domain $A$, values in $B$. Prove from the set axioms that if $A, B$ are sets, then $A \times B$ is a set.

Proof

$(a), (a, b) \subseteq P(A \cup B)$, so $(a, b) = \{(a), \{a, b\}\} \subseteq P(P(A \cup B))$. Therefore by the axiom of unordered pairs, the axiom of union to get $A \cup B = \cup\{A, B\}$, the power set axiom twice, and then subset construction, we get

$$A \times B = \{z \in P(P(A \cup B))| \text{ there exists an } a \in A \text{ and } b \in B \text{ such that } z = (a, b)\}$$

is a set.

3) a) Define the phrase "$A$ is a countable set".

$A$ is countable if there is a function $f$ with domain $\omega$ such that $f(\omega) = A$.

We write $f(n) = a_n$ and speak of the sequence $a_n$. Thus a set is countable if it can be written as a simple sequence.

b) Show that the Cartesian product $A \times B$ of countable sets is countable.

We have done this in class. We will not write a picture here. Let $A = \{a_0, a_1, \ldots\}, B = \{b_0, b_1, \ldots\}$. We order all pairs $(a_m, a_n)$

- Put $(a_m, a_n)$ before $(a_p, a_q)$ whenever $max(m, n) < max(p, q)$.
- Put $(a_m, a_n)$ before $(a_p, a_q)$ in case $max(m, n) = max(p, q)$ and $n < q$.
- Put $(a_m, a_n)$ before $(a_p, a_q)$ in case $max(m, n) = max(p, q)$ and $n = q$ and $m < p$.

4) a) State the definition by induction of exponentiation $a^x$ for $a, x \in \omega = \{0, 1, \ldots\}$

- $a^0 = 1$
- $a^{x+1} = (a^x)a$

b) Prove by induction that $a^{x+y} = (a^x)(a^y)$. Assume whatever properties of addition and multiplication you need. State the base step, inductive hypothesis and inductive conclusion to be proved clearly and completely.

Base step: $a^{0+1} = a^0a = 1a = a^0a^1$

Inductive Hypothesis: Suppose $a^{x+y} = a^x a^y$.

Inductive Conclusion: Prove that $a^{x+(y+1)} = a^x a^{y+1}$

Proof $a^{x+(y+1)} = a^{(x+y)+1} = a^{x+y}a$. By the inductive hypothesis, this is $(a^x a^y)a$. But $(a^x a^y)a = a^x(a^ya)$, and $a^ya$ is $a^{y+1}$ by definition. So finally

$a^{x+(y+1)} = a^x a^{y+1}$.

5 a) Define: $f : A \rightarrow B$ is onto.

b) If $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions, and both are onto, prove that $g \circ f$ is onto.
Proof. For each \( c \in C \), since \( g \) is onto, there is a \( b \in B \) with \( g(b) = c \). For that \( b \), since \( f \) is onto, there is an \( a \in A \), \( f(a) = b \). Then \( g(f(a)) = g(b) = c \). So \( g \circ f \) is onto.

6) Here is a form of the axiom of choice. If \( \{ A_b \}_{b \in B} \) is any indexed family of non-empty sets, indexed by set \( B \), there is a function \( g \) with domain \( B \) such that for all \( b \in B \), \( g(b) \in A_b \). If \( f : A \longrightarrow B \) is onto, use this form of the axiom of choice to prove that there is a \( g : B \longrightarrow A \) such that for all \( a \in A \), \( g(f(a)) = a \).

Proof. Let the index set be \( B \). For each \( b \in B \), let \( A_b = \{ a \in A \mid f(a) = b \} \). Let \( g \) be the axiom of choice function with domain \( B \) such that for each \( b \in B \), \( g(b) \in A_b \).