Directions:

1. You have 50 minutes to complete the exam.

2. The exam is closed book: no notes, calculators, or other aids allowed.

3. Read each question carefully to ensure you are answering what is asked.

4. You must show all of your work to receive credit.

5. Your work must be organized and legible to receive full credit.

6. You do not need to simplify the arithmetic in your answers completely.

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Problem 1. (10 pts.)

$n$ independent trials which result in a success with probability $p$ and a failure with probability $1 - p$ are performed. Denote by $P_n$ the probability that the $n$ trials result in an even number of successes (0 being considered an even number).

a) Show that

$$P_n = p(1 - P_{n-1}) + (1 - p)P_{n-1}.$$ 

Condition on the outcome of the first trial. Let $E$ be the event that $n$ trials result in an even number of successes, and $F$ be the event that the first trial is a success. So $\mathbb{P}(F) = p$.

So

$$P_n = \mathbb{P}(E|F) \cdot \mathbb{P}(F) + \mathbb{P}(E|F^c) \cdot \mathbb{P}(F^c)$$

$$= (1 - P_{n-1})p + P_{n-1}(1 - p)$$

b) Use the formula from a) in order to prove (by induction) that

$$P_n = \frac{1 + (1 - 2p)^n}{2}.$$ 

Base case: for $n = 1$, $P_1 = \frac{1 + 1 - 2p}{2} = 1 - p = \mathbb{P}(1 \text{ trial, no successes})$

Assume the formula holds for $n$, so $P_n = \frac{1 + (1 - 2p)^n}{2}$.

Then

$$P_{n+1} = p(1 - P_n) + (1 - p)P_n$$

$$= p - p \left( \frac{1 + (1 - 2p)^n}{2} \right) + 1 + (1 - 2p)^n - p \left( \frac{1 + (1 - 2p)^n}{2} \right)$$

$$= \frac{2p - 2p - 2p(1 - 2p)^n + 1 + (1 - 2p)^n}{2}$$

$$= \frac{1 + (1 - 2p)^n(1 - 2p)}{2}$$

$$= \frac{1 + (1 - 2p)^{n+1}}{2}$$

So this holds for all $n$. 
Problem 2. (10 pts.)
An urn contains \( n \) balls labeled 1 through \( n \). We select \( m \) balls randomly in sequence, each time replacing the ball previously selected. Each ball is equally likely to be selected. Let \( X \) be the maximum label of the \( m \) chosen balls. Determine the p.m.f. (probability mass function) of \( X \).

Let \( X_i = \) the number of the \( i \)th ball selected for \( i = 1, 2, ..., m \).

Then \( \mathbb{P}[X \leq k] = \mathbb{P}[X_1 \leq k, X_2 \leq k, ..., X_m \leq k] \)

All of the events that \( (X_i \leq k) \) are independent, because of the replacement, and each event happens with probability \( \frac{k}{n} \). So \( \mathbb{P}[X \leq k] \) is \( \prod_{i=1}^{m} \mathbb{P}[X_i \leq k] = \left( \frac{k}{n} \right)^m \)

Then,

\[
P_k = \mathbb{P}[X = k] = \mathbb{P}[X \leq k] - \mathbb{P}[X \leq k - 1]
= \left( \frac{k}{n} \right)^m - \left( \frac{k - 1}{n} \right)^m
\]
Problem 3. (10 pts.)
Let $X$ be a geometric random variable with parameter $p > 0$. This means that the p.m.f of $X$ is given by
\[ p(k) = \mathbb{P}[X = k] = (1 - p)^{k-1}p, \text{ for } k = 1, 2, \ldots \]

a) Compute the probability that $X$ is an even number. (3 pts.)
\[
\mathbb{P}[X = 2k] = \sum_{k=1}^{\infty} (1-p)^{2k-1}p = (1-p)p \sum_{k=1}^{\infty} (1-p)^{2k-2} = (1-p)p \sum_{j=0}^{\infty} ((1-p)^2)^j = (1-p)p \left( \frac{1}{1-(1-p)^2} \right) = \frac{(1-p)p}{(1-(1-p))(1+(1-p))} = \frac{1-p}{2-p}
\]

b) Let $Y = \frac{1}{X}$. Compute the expectation of $Y$. (7 pts.)

Hint: it might help to use $\frac{a^k}{k} = \int_0^a x^{k-1}dx$ and interchange sum and integral.

Note: $\frac{1-p}{1} < 1$.
\[
\mathbb{E}[Y] = \sum_{k=1}^{\infty} \frac{1}{k} (1-p)^{k-1}p = \frac{p}{1-p} \sum_{k=1}^{\infty} \frac{(1-p)^k}{k} = \frac{p}{1-p} \int_0^{1-p} \left( \sum_{k=1}^{\infty} x^{k-1} \right) dx = \frac{p}{1-p} \int_0^{1-p} 1 - x \ dx = -\frac{p}{1-p} \ln(1-x)|_0^{1-p} = -\frac{p}{1-p} (\ln(p) - \ln(1)) = -\frac{p}{1-p} \ln(p)
\]
Problem 4. (10 pts.)
Let $X$ be a random variable with cumulative distribution function $F$ given by

$$F(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

a) Find the probability density function $f(x)$ of $X$. (3 pts.)

$$f(x) = F'(x) = \frac{1 + x - x}{(1 + x)^2} = \frac{1}{(1 + x)^2}, \text{ so}$$

$$f(x) = \begin{cases} \frac{1}{(1+x)^2}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

b) Compute $\mathbb{P}[2 < X < 4]$. (3 pts.)

$$\mathbb{P}[2 < X < 4] = F(4) - F(2) = \frac{4}{3} - \frac{2}{3} = \frac{2}{15}$$

b) Compute the expectation of the random variable $Y = (1 + X)^2 e^{-2X}$. (4 pts.)

$$E[Y] = \int_0^\infty (1 + x)^2 \cdot e^{-2x} \cdot f(x) \, dx = \int_0^\infty (1 + x)^2 \cdot e^{-2x} \cdot \frac{1}{(1 + x)^2} \, dx$$

$$= \int_0^\infty e^{-2x} \, dx = \left. -\frac{1}{2} (e^{-2x}) \right|_0^\infty$$

$$= -\frac{1}{2} (0 - 1) = \frac{1}{2}$$
Problem 5. (10 pts.)
Let $X$ be a random variable with exponential distribution of rate 3. Then the probability density function of $X$ is given by
\[
f(x) = \begin{cases} 
3e^{-3x}, & \text{if } x \geq 0 \\
0, & \text{if } x < 0.
\end{cases}
\]

a) Prove that the random variable $X$ is memoryless. (2 pts.)
We want to show that $P(X > s + t | X > s) = P(X > t)$ (for $s, t \geq 0$).

\[
F(x) = \int_0^x 3e^{-3a} da = -e^{-3a} \bigg|_0^x = 1 - e^{-3x}
\]
So $F(s + t) = 1 - e^{-3(s+t)}$ and $F(s) = 1 - e^{-3s}$.

\[
P(X > s + t | X > s) = \frac{P(X > s + t \cap X > s)}{P(X > s)} = \frac{P(X > s + t)}{P(X > s)} = \frac{1 - (s + t)}{1 - F(s)} = \frac{e^{-3(s+t)}}{e^{-3s}} = e^{-3t} = 1 - F(t) = P(X > t)
\]

b) Let $Y = 1 - e^{-3X}$. Compute the cumulative distribution function and the probability density function of $Y$. What distribution has $Y$? (8 pts.)
\[
F_Y(y) = P(Y \leq y) = P(1 - e^{-3X} \leq y) = P(-e^{-3X} \leq y - 1)
= P(-3X \geq \ln(1 - y)) = P(X \leq -\frac{1}{3}\ln(1 - y)) = F_X\left(-\frac{1}{3}\ln(1 - y)\right)
= 1 - e^{-3 - \frac{1}{3}\ln(1 - y)} = 1 - e^{\ln(1 - y)}
= 1 - (1 - y) = y
\]
So $F_Y(y) = y$ and $f_y(y) = 1$ on $(0, 1)$. $Y$ has uniform distribution on $(0, 1)$.