Prelim 1 Solutions

Problem 1. (15 pts.) An urn contains 9 red, 11 white, and 15 black balls. We select randomly 10 balls. All possible drawings (outcomes) are assumed to be equally likely. What is the probability that:

(a) 5 red balls, 3 white balls, and 2 black balls are selected? (3 pts.)

\[
\frac{\binom{9}{5} \binom{11}{3} \binom{15}{2}}{\binom{35}{10}}
\]

(b) at least 7 red balls are selected? (4 pts.)

\[
P(7 \text{ red balls}) + P(8 \text{ red balls}) + P(9 \text{ red balls}) = \sum_{j=0}^{2} \binom{9}{7+j} \binom{26}{3-j} \binom{35}{10}
\]

\[
= \binom{9}{7} \binom{26}{3} + \binom{9}{8} \binom{26}{2} + \binom{9}{9} \binom{26}{1} \binom{35}{10}
\]

Note: A number of people argued that you can count the ways to select at least 7 balls by first counting the number of ways to select 7 red balls, and then the number of ways to select 3 from the remaining balls (regardless of color). The issue here is that this overcounts some possibilities because we have already accounted for the 3 remaining balls if they are red, by counting the number of ways to select 7 red balls. Selecting 7 red balls and then selecting 3 other balls of any color are not independent events.

(You can see this easily in a small example, say, if you wanted to select at least 2 red balls from an urn containing 3 red and 3 white balls. Label each ball with a number, and then list the ways by the above method.)

(c) all selected balls have the same color? (4 pts.)

\[
P(\text{all red}) + P(\text{all white}) + P(\text{all black}) = \frac{0 + \binom{11}{10} + \binom{15}{10}}{\binom{35}{10}}
\]

(d) Given that no red balls are selected, what is the conditional probability that there are exactly 4 white balls along the 10 chosen? (4 pts.)

We’ve reduced the sample space to drawing from 11 white and 15 black balls.

\[
P(4 \text{ white and 6 black}|\text{no red balls drawn}) = \frac{\binom{11}{4} \binom{15}{6}}{\binom{26}{10}}
\]
Problem 2. (10 pts.) We have a die and a coin available. Roll first the die and denote by $i$ the outcome (the number which shows up). Then toss the coin exactly $i$ times. What is the probability that we get exactly two heads?

For $F_i = \{ \text{die shows face } i \}$, $E = \{ \text{exactly 2 heads} \}$

$$P(F_i) = \frac{1}{6} \text{ for } i = 1, \ldots, 6$$

$$P(E) = \sum_{i=1}^{6} P(E|F_i)P(F_i), \text{ and } P(E|F_i) = \binom{i}{2} \frac{1}{2^i}$$

So $P(E) = \frac{1}{6} \sum_{i=2}^{6} \binom{i}{2} \frac{1}{2^i}$

Problem 3. (10 pts.) Prostate cancer is the most common type of cancer found in males. Assume that 1% of males aged 60-70 have prostate cancer. A medical test which detects if a male has prostate cancer is performed and suppose that such a test is 98% effective. This means that a male with prostate cancer has a 98% chance of a positive test, while a male without has a 2% chance of testing positive. What is the probability that a man aged 60-70 has prostate cancer, given that he tested positive?

For $E = \{ \text{person has cancer} \}$, $F = \{ \text{person tests positive} \}$

$$P(E|F) = \frac{P(E) + P(F|E)}{P(F)} = \frac{P(E) + P(F|E)}{P(E)P(F|E) + P(E^C)P(F|E^C)}$$

$$.01 \cdot .98$$

$$.01 \cdot .98 + .99 \cdot .02$$

$$.98 = \frac{0.33}{296}$$
Problem 4. (15 pts.)

(a) For any two events $E$ and $F$, show that

$$P(E \cap F^C) = P(E) - P(E \cap F)$$

$$P(E) = P(E \cap (F \cup F^C)) = P((E \cap F) \cup (E \cap F^C)) = P(E \cap F) + P(E \cap F^C)$$

because $(E \cap F)$ and $(E \cap F^C)$ are disjoint

$$\rightarrow P(E \cap F^C) = P(E) - P(E \cap F)$$

(b) Let $E$ and $F$ be two mutually exclusive events. Show that

$$P(E|E \cup F) = \frac{P(E)}{P(E) + P(F)}$$

$$P(E|E \cup F) = \frac{P(E \cap (E \cup F))}{P(E \cup F)}$$

$$= \frac{P((E \cap E) \cup (E \cap F))}{P(E \cup F)}$$

$$= \frac{P(E) + P(E \cap F) - P(E \cap E \cap F)}{P(E) + P(F) - P(E \cap F)}$$

$$= \frac{P(E)}{P(E) + P(F)}$$

since $P(E \cap F) = 0$.

(c) Show that, for $n > 0$:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0$$

Using the binomial theorem,

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (1)^{n-i} = (1 - 1)^n = 0$$