Math 441 homework 11 solutions:

14G. If a matrix $A$ is symmetric, it means that if $a_{ij} = 1$, then $a_{ji} = 1$ also. That is, if a permutation matrix is symmetric and the permutation sends $i$ to $j$, then it must also send $j$ to $i$. This means that a cycle can have length at most two. A cycle of length two is determined uniquely by the two elements of the cycle, so a symmetric $n \times n$ permutation matrix is equivalent to a way to divide $n$ elements into sets of size one or two. Example 14.11 shows that the exponential generating function for this is $e^{x + \frac{1}{2} x^2}$.

Another approach is to use Theorem 14.2. A permutation matrix corresponds to a permutation $\sigma$. If the matrix is symmetric, then $\sigma(i) = j$ means that $\sigma(j) = i$. Thus, each cycle in the matrix has length at most 2. The set of permutations in $S_n$ with all cycles of size at most 2 can be formed by breaking $[n]$ into parts, and then applying a single cycle of length at most 2 to each part. There is one such cycle for a part of size 1 or 2, and no such cycles for a part of any other size. Therefore, the exponential generating function for what we do to each part is $N(x) = x + \frac{2}{n}$. By Theorem 14.2, the exponential generating function we seek is $e^{x + \frac{1}{2} x^2}$.

A third approach is to start by finding a recursion. As in the previous approaches, a symmetric matrix corresponds to a permutation with all cycles of length at most two. If we have a permutation $\sigma \in S_{n+1}$, then $n+1$ is in a cycle of length either 1 or 2. If it is in a cycle of length 1, then there are $a_n$ ways to fill out the rest of the permutation. If it is a cycle of length 2, then there are $n$ ways to pick the other element of the cycle, and then $a_{n-1}$ ways to pick the rest of the permutation. Therefore, $a_{n+1} = a_n + na_{n-1}$. We have the exponential generating function $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$. From this, we can compute

$$f'(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} (a_n + na_{n-1}) \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + x \sum_{n=1}^{\infty} a_{n-1} \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + x \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

$$= f(x) + xf(x).$$

We start the second summation on the third line at $n = 1$ because the $n = 0$ term is zero, as it has a factor of $n$. Thus, we have $f'(x) = (1 + x)f(x)$. If $f(x) = Ce^{g(x)}$, then $f'(x) = Cg'(x)e^{g(x)} = g'(x)f(x)$. We have $g'(x) = 1 + x$,
from which $g(x) = x + \frac{x^2}{2}$. Therefore, $f(x) = Ce^{x+\frac{x^2}{2}}$ for some constant $C$.

We can compute that $\frac{d}{dx}Ce^{x+\frac{x^2}{2}} = C(1 + x)e^{x+\frac{x^2}{2}}$. This evaluates to $C$ at $x = 0$, so the Taylor series for $Ce^{x+\frac{x^2}{2}}$ has a linear coefficient of $C$. Since $a_1 = 1$, the linear coefficient should be 1, and so $C = 1$. Therefore, $f(x) = e^{x+\frac{x^2}{2}}$

14H. Put the 1 in some arbitrary position and place the rest of the numbers consecutively from there. Let $a_i$ be the symbol 0 or 1 in the $i$-th position on the circle. Define a function $f : [2n] \to \mathbb{R}$ by

$$f(x) = \left(\sum_{i=1}^{x} a_i\right) - \frac{x}{2}$$

Since $f$ has a finite domain, it must attain some minimum value. Suppose that $f(y)$ is this minimum value. Relabel the circle by subtracting $y$ from our previous labels, and adding $2n$ when necessary to keep the numbers between 1 and $2n$.

Let $b_i$ be the symbol 0 or 1 in the $i$-th position with this new labeling, and define a function $g : [2n] \to \mathbb{R}$ by

$$g(x) = \left(\sum_{i=1}^{x} b_i\right) - \frac{x}{2}$$

If $x + y \leq 2n$, then by checking terms, we get $g(x) = f(x + y) - f(y) \geq 0$ because $f(y)$ is the minimum value of $f(x)$. If $x + y > 2n$, then we check which terms are in each sum to compute $g(x) = f(x + y - 2n) + f(2n) - f(y) = f(x + y - 2n) - f(y) \geq 0$, because $f(2n) = 0$ and $f(y)$ is the minimum value of $f(x)$.

14I. Let the number of ways that this can be done for $2(n-1)$ vertices be $a_n$. The number 1 must be adjacent to an even number or else there would be an odd number of points on each side of the chord containing 1, which would mean some two points on opposite sides of the chord would be themselves connected by a chord, which would intersect the chord containing 1. If there are $2(c-1)$ points on one side of the chord containing 1, then there are $2(n-c-1)$ points on the other side. These can have chords drawn among them in $a_c$ and $a_{n-c}$ ways, respectively. These can be chosen independently, so there are $a_c a_{n-c}$ ways to fill out the rest of the chords. We can get all ways to draw the chords by summing over all ways to pick the point to which 1 is adjacent, which is summing over $c$. We get

$$a_n = \sum_{c=1}^{n-1} a_c a_{n-c}.$$  

This is precisely the recursion of the Catalan numbers, (14.10). We can check that $a_1 = 1$, so the initial term matches the Catalan numbers, and so we get the formula of that sequence: $a_n = \frac{1}{n} \binom{2n-2}{n-1}$. If we wish a formula for $2n$ points, we want $a_{n+1} = \frac{1}{n+1} \binom{2n}{n}$.  


There seems to be some disagreement about whether a regular graph necessarily means a simple graph. I wrote up a solution assuming that it did, so I’m going to leave it that way.

Let the number of such graphs for \( n \) vertices be \( a_n \), and let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \). Vertex \( n \) is adjacent to two other vertices. Either these two vertices are adjacent or else they are not. If they are adjacent, then these two vertices together with \( n \) form a triangle, while the rest of the vertices form a regular graph of degree two on \( n - 3 \) vertices. There are \( \binom{n-1}{2} \) ways to pick the two vertices adjacent to \( n \) and \( a_{n-3} \) ways to pick the rest of the graph.

If the other two vertices are not adjacent, we could remove vertex \( n \) and make its two former neighbors adjacent to get a regular graph of degree 2 on \( n - 1 \) vertices. Each such graph has \( n - 1 \) edges, so it has \( n - 1 \) ways to break an edge and make its two vertices adjacent to \( n \) to get a regular graph of degree 2 on \( n \) vertices. Therefore, the number of graphs for which \( n \) is not part of a triangle is \( (n-1)a_{n-1} \).

We can add the graphs for which \( n \) is part of a triangle to those for which it is not to get \( a_n = (n-1)a_{n-1} + \binom{n-1}{2}a_{n-3} \). We use this to compute

\[
f'(x) = \sum_{n=1}^{\infty} a_n x^{n-1} \frac{x^n}{(n-1)!} = \sum_{n=3}^{\infty} a_n x^{n-1} \frac{x^n}{(n-1)!} = \sum_{n=1}^{\infty} \left( (n-1)a_{n-1} + \binom{n-1}{2}a_{n-3} \right) x^{n-1} \frac{x^n}{(n-1)!} \]

\[
= \sum_{n=1}^{\infty} a_n x^{n-1} \frac{x^n}{(n-2)!} + \sum_{n=3}^{\infty} \frac{1}{2}a_{n-3} x^{n-1} \frac{x^n}{(n-3)!} = x \sum_{n=2}^{\infty} a_n x^{n-1} \frac{x^n}{(n-1)!} + \frac{1}{2} x^2 \sum_{n=0}^{\infty} a_n x^n \frac{x^n}{n!} \]

\[
= x \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!} + \frac{1}{2} x^2 \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = x f'(x) + \frac{1}{2} x^2 f(x). \]

Some lines are consequences of \( a_1 = a_2 = 0 \), as there are no regular graphs of degree 2 on one or two vertices. We can rearrange to get \((x-1)f'(x) = -\frac{1}{2} x^2 f(x)\), or \(f'(x) = -\frac{1}{2} x^2 f(x) = (x - \frac{1}{2} x - \frac{1}{2} x^2) f(x)\). If \( f(x) = e^{g(x)} \), then \( f'(x) = e^{g(x)} g'(x) \). We have \( g'(x) = x - \frac{1}{2} x - \frac{1}{2} x^2 \), so \( g(x) = -\frac{1}{4} x^2 + \frac{1}{2} x - \frac{1}{2} \).
\[ \frac{1}{2}x - \frac{1}{2} \ln |x - 1| + C. \] Hence,

\[
\begin{align*}
f(x) &= e^{-\frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{2} \ln |x - 1| + C} \\
&= e^{-\frac{x^2}{4} - \frac{x}{2} - \frac{1}{2} \ln |x - 1|} e^C \\
&= e^C e^{-\frac{x^2}{4} - \frac{x}{2} |x - 1|^{-\frac{1}{2}}} \\
&= e^C \frac{e^{-\frac{x^2}{4} - \frac{x}{2}}}{\sqrt{|x - 1|}}.
\end{align*}
\]

We can plug \( x = 0 \) into the definition of an exponential generating function and get \( f(0) = a_0 = 1 \) (since there is one graph on no vertices, and every vertex of it has degree 2). Above, we get \( f(0) = e^C \frac{e^0}{\sqrt{|0 - 1|}} = e^C \), so \( e^C = 1 \). Therefore,

\[
f(x) = e^{-\frac{x^2}{4} - \frac{x}{2}} \frac{1}{\sqrt{|x - 1|}}.
\]

Near \( x = 0 \), this is equivalent to

\[
f(x) = e^{-\frac{x^2}{4} - \frac{x}{2}} \frac{1}{\sqrt{1 - x}}.
\]

Alternatively, we could use Theorem 14.2. The graph is a disjoint union of cycles of length at least 3. This is done by breaking the vertices into parts and then imposing a cycle of length at least 3 on each part. If we have \( k \) vertices in a part and wish to impose an ordered cycle on them, then we can pick an arbitrary first vertex. There are \( k - 1 \) ways to pick which vertex this goes to along the cycle, then \( k - 2 \) ways to pick where the cycle goes from the next vertex, and so forth. The last vertex must go back to the first vertex, and there is 1 way to do this. Therefore, the number of ordered cycles on \( k \) vertices is \( (k - 1)(k - 2) \ldots (2)(1)(1) = (k - 1)! \). We want unordered cycles, and an unordered cycle corresponds to two ordered cycles, as an ordered cycle is an unordered one together with an orientation, that is, a choice of direction in which to traverse the cycle. Therefore, the number of cycles on \( k \) labeled vertices is \( \frac{1}{2}(k - 1)! \).

We need to have at least three vertices to make a cycle. Therefore, the exponential generating function for the number of ways to apply a cycle a set
of \( n \) labeled vertices is

\[
N(x) = \sum_{n=3}^{\infty} \frac{1}{2(n-1)!} \frac{x^n}{n!} = \frac{1}{2} \sum_{n=3}^{\infty} \frac{x^n}{n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^n}{n} = -\frac{1}{2} \ln(1-x) - \frac{x}{2} - \frac{x^2}{4}.
\]

By Theorem 14.2, the exponential generating function we are looking for is

\[
e^{N(x)} = e^{-\frac{1}{2} \ln(1-x) - \frac{x}{2} - \frac{x^2}{4}} = e^{-\frac{1}{2} \ln(1-x)} e^{-\frac{x}{2} - \frac{x^2}{4}} = (1-x)^{-\frac{1}{2}} e^{-\frac{x}{2} - \frac{x^2}{4}} = e^{-\frac{x}{2} - \frac{x^2}{4}} = \sqrt{\frac{1}{1-x}}.
\]

If we allow loops and multiple edges, then the number of ways to apply a cycle to a set of 1 or 2 vertices is now 1 = 0! = 1!. Thus, we would get

\[
N(x) = \sum_{n=1}^{\infty} \frac{1}{2(n-1)!} \frac{x^n}{n!} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^n}{n} = -\frac{1}{2} \ln(1-x).
\]

Our answer would then be

\[
e^{N(x)} = e^{-\frac{1}{2} \ln(1-x)} = \frac{1}{\sqrt{1-x}}.
\]

14N. Let \( a_n \) be the number of walks from \((0,0)\) to \((n,n)\) that never go above the line \( x = y \). Note that in order for a walk to never go above or touch the line \( x = y \) except at \((0,0)\) and \((n,n)\), if \( n \geq 2 \), it must start by going right one step, then follow the same sequence of steps as a walk from \((0,0)\) to \((n-1,n-1)\) that never goes above the line \( x = y \), and then go up one step. Conversely, any such walk goes from \((0,0)\) to \((n,n)\) without going above or touching the line \( x = y \) except at the endpoints. There are \( a_{n-1} \) such walks, so there are \( a_{n-1} \) ways to get from \((0,0)\) to \((n,n)\) without touching the line \( x = y \) except at the endpoints.
Let \( a_n^i \) be the number of walks from \((0,0)\) to \((n,n)\) such that the first time the walk touches the line \(x = y\) other than at \((0,0)\) comes at \((i,i)\). If the first step is to go diagonally up and right, then say \(i = 0\). Clearly \( a_n = \sum_{i=0}^{n} a_n^i \). If \(i > 0\), then a walk that first touches the line \(x = y\) at \((i,i)\) is a walk from \((0,0)\) to \((i,i)\) that never goes above or touches the line \(x = y\) except at the endpoints, and then a walk from \((i,i)\) to \((n,n)\) that never goes above the line \(x = y\). There are \(a_{i-1}\) ways to pick the former walk and \(a_{n-i}\) ways to pick the latter, and these can be chosen independently. Therefore, \(a_n^i = a_{i-1}a_{n-i}\).

If \(i = 0\), then we start with a diagonal step. The number of ways to finish the path is the number of paths from \((1,1)\) to \((n,n)\) that do not go above the line \(x = y\). This is equivalent to the number of such paths from \((0,0)\) to \((n-1,n-1)\), which is \(a_{n-1}\). Therefore, \(a_n^0 = a_{n-1}\). Substitute this to get our recursion of

\[
a_n = \sum_{i=0}^{n} a_n^i = a_{n-1} + \sum_{i=1}^{n} a_{i-1}a_{n-i}
\]

for \(n \geq 1\).