Math 441 homework 10 solutions:

13A. Pick one distinguished position on the circle. Any arrangement of \( r \) numbers on the circle is equivalent by rotation to one without the distinguished position filled. There is a bijection between ways to put \( r \) numbers on a circle in \( n-r \) positions with a distinguished position not filled and ways to put \( r \) numbers on a circle in \( n \) positions with no two adjacent, by adding a blank spot immediately clockwise of each used position. There are \( \binom{n-r}{r} \) ways to do the latter, but they come in sets of \( n-r \) placements that are equivalent by rotation. Hence, there are \( \frac{1}{n-r} \binom{n-r}{r} \) ways to choose the positions of the numbers. There are \( r \) ways to pick where to put the 1, so there are \( r \) ways to place the numbers in given positions, and hence \( \frac{1}{n-r} \binom{n-r}{r} \binom{n-r-1}{r-1} \) ways to arrange the numbers.

Alternatively, we can fix the position of the 1 on the circle, since there is a unique way to rotate any arrangement that puts the 1 in the desired position. The two spots adjacent to the 1 must be empty. This leaves \( n-3 \) positions in a line that must hold \( r-1 \) remaining numbers. By Example 13.1, there are \( \binom{(n-3)-(r-1)+1}{r-1} \) ways to do this. Having the numbers in order means that the remaining numbers are uniquely determined by the positions chosen.

13C. Each number can be in \( A_1 \), in \( A_2 \), or in neither. These three options can be chosen independently for each number, so there are \( 3^n \) possibilities.

Alternatively, we could let \( E_i \) be the event that \( i \in A_1 \cap A_2 \). Being in \( E_i \) forces both \( i \in A_1 \) and \( i \in A_2 \). We can compute \( |E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}| = 4^{n-k} \), as there are \( n-k \) values of \( j \) for which we have no restrictions on whether \( j \) is in \( A_1 \) or \( A_2 \), and four ways to pick a combination of which sets it is in. There are \( \binom{n}{k} \) ways to pick \( k \) elements of \([n]\), so we can use inclusion-exclusion to get

\[
\left| \bigcap_{i=1}^{n} E_i^c \right| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} 4^{n-k}.
\]

By the binomial theorem, this is the binomial expansion for \((4-1)^n\), so the answer is \( 3^n \).

13D. Half of the possible \( k \)-tuples have 1 ∈ \( A_1 \), half have 1 ∈ \( A_2 \), and so forth, and these are independent. Hence, 1 − (.5)^k of \( k \)-tuples have a 1 in the union. There are \( k \) sets, with \( 2^n \) possibilities for each, so there are \( 2^k \) possible \( k \)-tuples. Hence, 1 contributes \( 2^{nk} - 2^{(n-1)k} \) to the sum. There are \( n \) possible elements of the set, each of which contributes the same to the sum by symmetry, so the sum is \( n(2^{nk} - 2^{(n-1)k}) \).

13F. We use induction on \( n \). If \( n = 2 \), there are only two permutations, one of which has an odd number of cycles and one an even number of cycles.

The number of permutations in \( S_n \) for which \( n \) is a 1-cycle with an even
number of cycles is the same as the number of permutations in \( S_{n-1} \) with an odd number of cycles, by adding the single cycle \((n)\) to a permutation on \( S_{n-1} \). Likewise, the number of such permutations of \( S_n \) with an odd number of cycles is the number of permutations in \( S_{n-1} \) with an even number of cycles. By the inductive hypothesis, \( S_{n-1} \) has just as many permutations with an even number of cycles as an odd number of cycles, so \( S_n \) has just as many of each for which \((n)\) is a cycle by itself.

Likewise, each permutation in \( S_{n-1} \) with an even number of cycles corresponds to \( n-1 \) cycles in \( S_n \) for which \( n \) is not a 1-cycle and there are an even number of cycles by inserting \( n \) into another cycle in an arbitrary spot. Similarly, each permutation in \( S_{n-1} \) with an odd number of cycles corresponds to \( n-1 \) cycles in \( S_n \) for which \( n \) is not a 1-cycle and there is an odd number of total cycles. By the inductive hypothesis, \( S_{n-1} \) has just as many permutations with an even number of cycles as an odd number of cycles, so \( S_n \) has just as many permutations for which \( n \) is not a 1-cycle and an odd number of cycles as an even number of cycles.

The total number of permutations in \( S_n \) with an even number of cycles is the number with an even number of cycles for which \( n \) is a 1-cycle by itself plus those for which \( n \) is not a 1-cycle. The same is true for those with an odd number of cycles, and we have shown that there are just as many with an even number of cycles as an odd number of cycles in each type, so there are the same number of total permutations in \( S_n \) with an even number of cycles as an odd number of cycles.

13K. In order to partition \( n+2 \) numbers into \( n \) sets, either we have two sets of size two and \( n-2 \) sets of size one or else we have one set of size three and \( n-1 \) sets of size one. For the latter, there are \( \binom{n+2}{3} \) ways to pick which three numbers go in the same set. For the former, there are \( \binom{n+2}{2} \) ways to pick the two elements of one set of size two and then \( \binom{2}{2} \) ways to pick the elements of the other. We could have chosen the same sets in the other order, so there are \( \frac{1}{2} \binom{n+2}{3} \binom{2}{2} \) such partitions. Therefore, we have

\[
S(n+2, n) = \binom{n+2}{3} + \frac{1}{2} \binom{n+2}{2} \binom{n}{2} \\
= \binom{n+2}{3} + \frac{(n+2)(n+1)n(n-1)}{(2)(2)(2)} \\
= \binom{n+2}{3} + 3 \binom{n+2}{4}.
\]

We know that \((1-x)^{-1} = \sum_{n=0}^{\infty} x^n\). If we take the fourth derivative of both sides of this, we get

\[
\frac{4!}{(1-x)^5} = \sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)x^{n-4}
\]
\[
\frac{x(1 + 2x)}{(1 - x)^5} = (x + 2x^2) \sum_{n=0}^{\infty} \binom{n + 4}{4} x^n
\]
\[
= \sum_{n=0}^{\infty} \binom{n + 4}{4} x^{n+1} + 2 \sum_{n=0}^{\infty} \binom{n + 4}{4} x^{n+2}
\]
\[
= \sum_{n=0}^{\infty} \binom{n + 3}{4} x^n + 2 \sum_{n=0}^{\infty} \binom{n + 2}{4} x^n
\]
\[
= \sum_{n=0}^{\infty} \left( \binom{n + 3}{4} + 2 \binom{n + 2}{4} \right) x^n
\]
\[
= \sum_{n=0}^{\infty} \left( \binom{n + 2}{3} + 3 \binom{n + 2}{4} \right) x^n
\]
\[
= \sum_{n=0}^{\infty} S(n + 2, n)x^n
\]