1. CLO, page 53, Problem 5.

2. CLO, page 73, Problem 3.

3. Suppose that $k$ is an infinite field. Let $X \subset k^3$ be the set \{(t, t^2, t^3) : t \in k\}.
   
   (a) Use the parametrization to show that $z^2 - x^4y$ vanishes at every point of $X$.
   
   (b) Find a representation $z^2 - x^4y = h_1(y - x^2) + h_2(z - x^3)$, where $h_1, h_2 \in k[x, y, z]$.
   
   (c) Use the division algorithm to show that $I(X) = \langle y - x^2, z - x^3 \rangle$.

4. Let $M$ be an $m \times n$ matrix with non-negative real entries, and let $r_1, \ldots, r_m$ denote the rows of $M$. Assume that $\ker M \cap \mathbb{Z}^n = (0)$, that is, the only solution to $Mx = 0$, where $x$ is a $n \times 1$ column vector with all integer entries, is the zero vector.

   Define a binary relation $>_M$ on the monomials in $R = k[x_1, \ldots, x_n]$ as follows: $x^a >_M x^b$ if there is an $\ell \leq m$ such that $a \cdot r_i = b \cdot r_i$, for $1 \leq i \leq \ell - 1$, and $a \cdot r_\ell > b \cdot r_\ell$.

   (a) Show that $>_M$ is a monomial ordering on (the monomials of) $R$.
   
   (b) If 
   
   $$ M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, $$
   
   show that $>_M$ equals $>_\text{grevlex}$ on $R = k[x, y, z]$.
   
   (c) If $M$ is the $n \times n$ identity matrix, then $>_M$ equals $>_\text{lex}$.

5. Fix the lexicographic order on $R = k[x_1, \ldots, x_n]$, with $x_1 > x_2 > \ldots > x_n$. Let $A = (a_{ij})$ be an $m \times n$ matrix with entries in $k$, and let

   $$ f_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n $$

   be the linear polynomials in $R$ determined by the rows of $A$. Suppose that $B = (b_{ij})$ is the row-reduced echelon matrix determined by $A$, and let $g_1, \ldots, g_r$ be the linear polynomials in $R$ determined by the non-zero rows of $B$.

   (a) Prove that $\langle f_1, \ldots, f_m \rangle = \langle g_1, \ldots, g_r \rangle$.
   
   (b) Show that $G = \{g_1, \ldots, g_r\}$ is a Groebner basis of $\langle f_1, \ldots, f_m \rangle$.

6. How long did you spend on this problem set? And did you find it (a) too challenging, (b) just right, or (c) too easy?