1. (a) Show that \( X = \{(x, x) : x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2 \) is not an affine variety. (Hint: if \( f \in \mathbb{R}[x, y] \) vanishes on \( X \), prove that \( f(1, 1) = 0 \). It might be useful to consider \( g(t) = f(t, t) \).
(b) Show that \( Y = \{(x, y) \in \mathbb{R}^2 : y > 0\} \subset \mathbb{R}^2 \) is not an affine variety.

2. Consider the set \( U(1) := \{z \in \mathbb{C} : zz = 1\} \).
(a) If we identify \( \mathbb{C} \) with \( \mathbb{R}^2 \), then show that \( U(1) \subset \mathbb{R}^2 \) is an affine variety. What is it?
(b) Prove that \( U(1) \subset \mathbb{C} \) is not an affine variety in \( \mathbb{C}^1 \).

3. Consider the map defined by \( f : k^3 \to k^6 \), defined by
\[
(x, y, z) \mapsto (a, b, c, d, e, f) = (x^2, xy, xz, y^2, yz, z^2).
\]
Let \( I \subset k[a, b, c, d, e, f] \) be the ideal generated by all of the \( 2 \times 2 \) minors of the matrix
\[
\begin{pmatrix}
a & b & c \\
b & d & e \\
c & e & f
\end{pmatrix}.
\]
Show that the image of \( f \) is contained in the affine variety \( \mathbb{V}(I) \).
Extra credit: is the map onto? i.e. for every point \( p \in \mathbb{V}(I) \), can you find \( (x, y, z) \) such that \( f(x, y, z) = p \)? Either find a point not in the image, or prove that the map is onto.

4. Let \( \mathbb{F}_2 \) be the finite field with 2 elements, defined in the last homework. Let \( I \subset \mathbb{F}_2[x, y, z] \) be the ideal of all polynomials which vanish at all 8 points of \( \mathbb{F}^3 \).
(a) Show that \( \langle x^2 - x, y^2 - y, z^2 - z \rangle \subset I \).
(b) For constants \( a_0, a_1, \ldots, a_7 \in \mathbb{F} \), show that the polynomial
\[
a_0xyz + a_1xy + a_2xz + a_3yz + a_4x + a_5y + a_6z + a_7
\]
is in \( I \) if and only if \( a_0 = a_1 = \cdots = a_7 = 0 \).
(c) Show that \( \langle x^2 - x, y^2 - y, z^2 - z \rangle = I \). (The division algorithm could be useful here).

5. (a) Suppose \( I = \langle x^a \rangle \), \( J = \langle x^b \rangle \) are monomial ideals each generated by one element, where \( a, b \in \mathbb{N}^n \). Show that \( I \cap J = \langle x^c \rangle \), for some \( c \). What is \( c \)? We denote \( c = \text{lcm}(a, b) \).
(b) Find (generators for) \( \langle x \rangle \cap \langle x^2, xy, y^2 \rangle \subset k[x, y] \).
(c) Now suppose that \( I = \langle x^{a(1)}, \ldots, x^{a(r)} \rangle \) and \( J = \langle x^{b(1)}, \ldots, x^{b(s)} \rangle \) are monomial ideals in \( R = k[x_1, \ldots, x_n] \), where \( a(i), b(j) \in \mathbb{N}^n \), for all \( i \) and \( j \). Show that \( I \cap J \) is a monomial ideal, generated by the monomials \( x^{\text{lcm}(a(i), b(j))} \), for \( i = 1..r, j = 1..s \). (Notice this is the lcm defined in part (a), not the lcm of integers!) (Later we will see that this is very special behavior for ideals: intersections of more general ideals are harder to compute).

6. How long did you spend on this problem set? And did you find it (a) too challenging, (b) just right, or (c) too easy?