(11.1) Let $G = D_6$ and $H = \langle r \rangle = \{e, r, r^2, r^3, r^4, r^5\}$. Since $s \not\in H$, we compute $sH = \{s, sr, sr^2, sr^3, sr^4, sr^5\}$. We see that $H$ and $sH$ are disjoint and $H \cup sH = D_6$.

Let $G = D_6$ and $H = \langle r^3 \rangle = \{e, r^3\}$. Since $r \not\in H$, we compute $rH = \{r, r^4\}$. We see that $r^2 \not\in H$ and $rH$, then $r^2H = \{r^2, r^5\}$. Similarly, we compute $sH = \{s, sr^3\}$, $srH = \{sr, sr^4\}$, $sr^2H = \{sr^2, sr^5\}$. Note that all these sets are disjoint and their union gives $D_6$.

Let $G = A_4 = \{id, (12)(34), (13)(24), (14)(23), (123), (142), (134), (143), (234), (243)\}$ and $H = \langle (234) \rangle = \{id, (234), (243)\}$. Since $(123) \not\in H$, we have $(123)H = \{(123), (12)(34), (124)\}$. Finally, $(142)H = \{(142), (14)(23), (143)\}$. Observe that all these sets are disjoint and their union gives $A_4$.

(11.2) Assume $g_1H = g_2H$. Then $g_2 = g_2 e \in g_2H = g_1H$ thus $g_2 = g_1 h$ for some $h \in H$. That gives $g_1^{-1} g_2 = h \in H$ as needed.

Now assume $g_1^{-1} g_2 = h \in H$. For any $g_1x \in g_1H$, we have

$$g_1x = g_2 g_2^{-1} g_1 x = g_2 (g_1^{-1} g_2)^{-1} x = g_2 (h^{-1} x) \in g_2 H,$$

so $g_1H \subset g_2 H$.

Similarly, for any $g_2y \in g_2H$, we have $g_2y = g_1hy \in g_1H$, so $g_2H \subset g_1H$. This gives $g_1H = g_2H$.

(11.3) $H, K$ are subgroups of $G$ with $|H| = a$, $|K| = b$, and $\gcd(a, b) = 1$. For any $x \in K \cap H$ the order of $x$ is finite, since $K, H$ are finite subgroups. Let $n$ be the order of $x$. From Lagrange’s Theorem we get that $n$ divides both $a$ and $b$. Therefore $n = 1$ and $x = e$. 
(11.4) Let $G$ be a group with $|G| = pq$ for some primes $p,q$. Let $H$ be a proper nontrivial subgroup of $G$. Then by Lagrange’s Theorem $|H|$ is either $p$ or $q$ since $p$ and $q$ are primes. By Corollary (11.3), $H$ is cyclic.

(11.8) Does $A_5$ have a subgroup of every order dividing 60?

No. For example, $A_5$ does not have a subgroup of order 30 or a subgroup of order 15. Assume that $A_5$ has a subgroup $H$ satisfying $|H| = 30$. Then $H$ cannot contain all 3-cycles since we know that $A_5$ is generated by 3-cycles. So, there is at least one 3-cycle $\sigma$ which is not in $H$. Then consider $H, \sigma H$, and $\sigma^2 H$. Then these left cosets cannot all be disjoint from one another, because they each have 30 elements and $A_5$ has 60 elements. Now we will check that no matter which pair of these has nonempty intersection, we can always deduce that $\sigma \in H$. Assume $H$ and $\sigma H$ intersect, then there is an $h$ in $H \cap \sigma H$. Since $h$ is also in $\sigma H$, we can write it as $h = \sigma h_1$ for some $h_1 \in H$. This implies $\sigma = hh_1^{-1} \in H$ (verify that we get $\sigma \in H$ in other cases). We get a contradiction since we assumed that $\sigma \notin H$. Hence, $A_5$ cannot have a subgroup of order 30.

The case of order 15 is much more complicated, as the following discussion shows. Suppose we have a subgroup $H$ of $A_5$ of order 15, and let’s see what could go wrong. By Lagrange’s theorem, every non-identity element would have to have order 3 or 5. The only elements of $S_5$ which have orders 3 or 5 are 3-cycles and 5-cycles. If all the elements of $H$ were 5-cycles, then we would have $15 \equiv 1 \pmod{4}$, since each subgroup generated by a 5-cycle has 4 non-identity elements. So $H$ has to have a 3-cycle $(abc)$. The product of two 3-cycles $(abc)(xyz)$ is a 5-cycle if $\#(\{a, b, c\} \cap \{x, y, z\}) = 1$, and the subgroup generated by them has an element of order 2 if $\#(\{a, b, c\} \cap \{x, y, z\}) = 2$:

\[
(abc)(cde) = (abcde)
\]

\[
(abc)(bcd) = (ab)(cd)
\]

\[
(abc)(cbd)^2 = (ab)(cd)
\]

Since $H$ doesn’t have any elements of order 2, the most number of 3-cycles it can contain is four: $(abc), (abc) = (abc)^2, (ade)$ and $(ade) = (ade)^2$ for some permutation $(abcde)$ of $(12345)$. But in fact, it can’t contain four, because the rest of the elements would have to be 5-cycles, and 11 is not congruent to 1 mod 4. So it can contain at most two 3-cycles, so must contain exactly two 3-cycles, say $(123)$ and $(132) = (123)^2$, and three 5-cycles which generate distinct subgroups. We can write each of these 5-cycles starting with 1, and by replacing it with a power, we may assume 1 goes to 3. So we may assume our generators are three of the following:

1. $(13245)$
2. $(13254)$

2
3. (13425)
4. (13452)
5. (13524)
6. (13523).

In each of these cases $(13xyz)(123)$ has 3 as a fixed point, so is not a 5-cycle, is not $(123)$ and is not $(132)$. So it's either a different 3-cycle or it has order 2, both of which are impossible in $H$. This concludes the discussion that $A_5$ cannot have a subgroup of order 15.