The t-tests and confidence intervals

A summary of concepts from basic statistics. In the textbook, this can be found in sections 7.2, 7.4, 7.5 and 8.5.

The \( \chi^2 \) (chi square) distribution (sec 7.2, p. 393)

If the random variables \( X_1, \ldots, X_n \) are i.i.d. and if each of these variables has a standard normal distribution, then the sum of squares \( X_1^2 + \ldots + X_n^2 \) has a \( \chi^2 \) distribution with \( n \) degrees of freedom (Theorem 7.2.2).

The t distribution (sec 7.4, p. 404)

Consider two independent random variables \( Y \) and \( Z \), such that \( Y \) has a \( \chi^2 \) distribution with \( n \) degrees of freedom and \( Z \) has a standard normal distribution. Suppose that a random variable \( X \) is defined by the equation

\[
X = \frac{Z}{(Y/n)^{1/2}}.
\]

Then the distribution of \( X \) is called the t distribution with \( n \) degrees of freedom (7.4.1.).

Relation to normal samples: Let \( X_1, \ldots, X_n \) be a random sample from a normal distribution having mean \( \mu \) and variance \( \sigma^2 \). Let \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \) and

\[
\sigma' = \left( \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right)^{1/2}.
\]

Then the distribution of

\[
U = \frac{n^{1/2} (\bar{X}_n - \mu)}{\sigma'}
\]

is a t distribution with \( n - 1 \) degrees of freedom (7.4.6).

Confidence interval for the mean of a normal distribution (sec. 7.5)

We assume that \( X_1, \ldots, X_n \) form a random sample from a normal distribution with unknown mean \( \mu \) and and unknown variance \( \sigma^2 \). Let \( t_{n-1}(x) \) denote the p.d.f. of the t distribution with \( n - 1 \) degrees of freedom, and let \( c \) be a constant such that

\[
\int_{-c}^{c} t_{n-1}(x)dx = \gamma. \quad (2)
\]

For every value of \( n \), the value of \( c \) can be found from the table of the t distribution given at the end of the textbook. For example, if \( n = 12 \) and if \( T_{11}(x) \) denotes the d.f. of the t distribution with 11 degrees of freedom, then

\[
\int_{-c}^{c} t_{11}(x)dx = T_{11}(c) - T_{11}(-c) = T_{11}(c) - [1 - T_{11}(c)]
= 2T_{11}(c) - 1. \quad (3)
\]
If $\gamma = 0.95$, it follows from Eq. (2) that $T_{11}(c) = 0.975$. It is found from the table that $c = 2.201$, the 0.975 quantile of the t distribution with 11 degrees of freedom.

Because the random variable $U$ defined by Eq. (1) has a t distribution with $n - 1$ degrees of freedom, Eq. (2) implies that $\Pr(-c < U < c) = \gamma$. This relation can be rewritten as follows:

$$\Pr(-c < U < c) = \Pr \left( -c < \frac{n^{1/2} (\bar{X}_n - \mu)}{\sigma'} < c \right)$$

$$= \Pr \left( -\frac{\sigma'}{n^{1/2}} < \bar{X}_n - \mu < \frac{\sigma'}{n^{1/2}} \right)$$

$$= \Pr \left( -\frac{\sigma'}{n^{1/2}} < \mu - \bar{X}_n < \frac{\sigma'}{n^{1/2}} \right)$$

$$= \Pr \left( \bar{X}_n - \frac{\sigma'}{n^{1/2}} < \mu < \bar{X}_n + \frac{\sigma'}{n^{1/2}} \right) = \gamma. \quad (4)$$

Thus, Eq. (4) states that regardless of the unknown value of $\sigma$, the probability is $\gamma$ that $\mu$ will lie between the random variables $A = \bar{X}_n - (\sigma / n^{1/2})$ and $B = \bar{X}_n + (\sigma / n^{1/2})$.

In a practical problem, Eq. (4) is applied as follows: After the values of the variables $X_1, \ldots, X_n$ in the random sample have been observed, the values of $A$ and $B$ are computed. If these values are $A = a$ and $B = b$, then the interval $(a, b)$ is called a confidence interval for $\mu$ with confidence coefficient $\gamma$ or a $100\gamma$ percent confidence interval for $\mu$. It is then common to make the statement that the unknown value of $\mu$ lies in the interval $(a, b)$ with confidence $\gamma$.

The t-test for hypotheses about the mean of a normal distribution when the variance is unknown

We again assume that $X_1, \ldots, X_n$ form a random sample from a normal distribution with unknown mean $\mu$ and unknown variance $\sigma^2$. We shall suppose now that the following hypotheses are to be tested:

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0. \quad (5)$$

Here the alternative hypothesis is two-sided. In example 8.1.8, p. 458 of the textbook it is shown how to derive a level $\alpha_0$ test from the confidence interval (4). The coefficient $\gamma$ confidence interval is the observed value of the interval

$$\left( \bar{X}_n - \frac{\sigma'}{n^{1/2}}, \bar{X}_n + \frac{\sigma'}{n^{1/2}} \right) \quad (6)$$

where $c$ is the $(1 + \gamma)/2$ quantile of the t distribution with $n - 1$ degrees of freedom. Indeed we have $2T_{n-1}(c) - 1 = \gamma$ from (3), hence

$$T_{n-1}(c) = (1 + \gamma)/2. \quad (7)$$

For each $\mu_0$, we can use the interval (6) to find a level $\alpha_0 = 1 - \gamma$ test of the hypotheses (5).
Construction of the test: The test will reject $H_0$ if $\mu_0$ is not in the interval (6).

It is obvious that $\mu_0$ is not in the interval if and only if the distance of $\mu_0$ from the center exceeds $1/2$ times the width of the interval, that is exceeds $c\sigma/n^{1/2}$. Thus the test rejects $H_0$ if and only if

$$|\bar{X}_n - \mu_0| \geq \frac{c\sigma}{n^{1/2}}$$

or equivalently if and only if

$$\left| \frac{n^{1/2} \bar{X}_n - \mu_0}{\sigma'} \right| \geq c.$$

Thus the form of the test is

"Reject $H_0$ if $\left| \frac{n^{1/2} \bar{X}_n - \mu_0}{\sigma'} \right| \geq T_{n-1}^{-1}(1 - \alpha_0/2)$"

where $T_{n-1}^{-1}$ is the quantile function of the t distribution with $n-1$ degrees of freedom. Indeed from (7) we have $c = T_{n-1}^{-1}((1 + \gamma)/2)$ and $\alpha_0 = 1 - \gamma$, hence $\gamma = 1 - \alpha_0$ and

$$c = T_{n-1}^{-1}(1 - \alpha_0/2).$$

Is this a level $\alpha_0$ test? We compute the probability of error of first kind. Assume $H_0$ is true and the test rejects, what is the probability of this? If $H_0$ is true then $\mu_0$ is the true mean. By (4)

$$\Pr \left( \bar{X}_n - \frac{c\sigma'}{n^{1/2}} < \mu_0 < \bar{X}_n + \frac{c\sigma'}{n^{1/2}} \right) = 1 - \alpha_0$$

hence for the contrary event

$$\Pr \left( \mu_0 \text{ is not in the interval } \left( \bar{X}_n - \frac{c\sigma'}{n^{1/2}}, \bar{X}_n + \frac{c\sigma'}{n^{1/2}} \right) \right) = \alpha_0.$$

By the construction of the test,

$$\Pr (\text{Test rejects } H_0) = \alpha_0$$

under the assumption that $H_0$ is true, so the test has level $\alpha_0$.

The one sided t-test. With a normal sample $X_1, \ldots, X_n$ as before, consider hypotheses

$$H_0 : \mu \leq \mu_0$$
$$H_1 : \mu > \mu_0.$$

The form of this test is

"Reject $H_0$ if $n^{1/2} \frac{\bar{X}_n - \mu_0}{\sigma'} \geq T_{n-1}^{-1}(1 - \alpha_0)$"

(see 8.5). It is also shown in the textbook (p. 487) that this is a level $\alpha_0$ test.

p-values. For given observed values of the sample, the $p$-value is the smallest $\alpha_0$ such that the test still rejects.