Totally Awesome Taxicab homework

1. Let $d_T$ denote the taxicab metric, and $d_E$ denote the usual Euclidean metric. One of the following inequalities holds for all $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{R}^2$. Which one?
   
   (a) $d_T(p, q) \leq d_E(p, q)$
   
   (b) $d_E(p, q) \leq d_T(p, q)$

2. Determine the taxicab radian angle measure of the angle shown below.

   ![Diagram of an angle in a taxicab plane]

3. Let $a$ and $b$ be real numbers. Show that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by
   
   $$f(x, y) = (x + a, y + b)$$

   is an isometry of the taxicab metric. Describe what the map $f$ does to $\mathbb{R}^2$.

4. A well known Euclidean trig identity is: $\sin^2 \theta + \cos^2 \theta = 1$.
   
   (a) Can you come up with an analogous trig identity for taxi-cab geometry using the functions $tsin_0 \theta$ and $tcos_0 \theta$.
   
   (b) Will your identity work for other reference angles $\phi$? That is, if you replace the $0$ with $\phi$, and write your identity with $tsin_\phi \theta$ and $tcos_\phi(\theta)$, is it still true?

5. When measured in the usual radians the sum of the angles in a Euclidean triangle is $\pi$. What can you say about the sum of the angles (in t-radians) in a taxicab triangle? Try a few triangles, and see if you can come up with a conjecture.
6. In Euclidean geometry, equilateral triangles are determined by their side length. Therefore, up to congruence, there is one triangle with all sides having length equal to one. In this question, we will be exploring equilateral taxicab triangles with side length equal to 1. In the figures below, we have the unit circle in the taxicab plane, along with triangle \( \triangle PQR \).

Note that in both figures, the triangle has point \( P \) at the origin, and the other two points, \( Q \) and \( R \) are on the unit circle. Therefore, the sides \( \overline{PQ} \) and \( \overline{PR} \) have length equal to 1.

(a) Find the coordinates of \( R \) so that \( \triangle PQR \) in Figure 1 (above left) is equilateral.

(b) Determine the taxicab radian measure of the angles of your triangle you found in part (a).

(c) Now consider a triangle situated such as the one in Figure 2 (above right); that is, a triangle with one point \( P \) at the origin, and two points \( Q \) and \( R \) on the unit circle in the first quadrant. Can you determine coordinates for \( Q \) and \( R \) so that \( \triangle PQR \) is equilateral?

(d) What are the angle measures of the triangles you found in part (c).

(e) Are the triangles you found in parts (a) and (c) congruent?

(f) How many non-congruent equilateral triangles of the type in Figure 2 are there? Can you describe them?

7. Can you generalize the taxicab (or \( \ell_1 \)) metric to \( \mathbb{R}^3 \)? What would the formula be? What would the unit sphere look like?