Note Taker Checklist Form - MSRI

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Talk Title and Workshop assigned to:
Kanumorphisms to mapping class groups
Intro to Geometric Group Theory

Lecturer (Full name): Daniele Croves

Date & Time of Event: 2/27 3:45 - 4:35 pm

Check List:
( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be
the note taker, and that you will need to make copies of their own notes, if
any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc).
This can be done either before the lecture is to begin or after the lecture;
please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be
provided by Computing Staff) – Scan this sheet first, then materials. In the
subject heading, enter the name of the speaker and date of their talk.

Please do NOT use pencil or colored pens other than black when taking notes as the
scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: Mapping Class, group, homomorphisms,
   Splittings, curve complex

2. Please summarize the lecture in 5 or less sentences.

   The purpose of the talk was to state & discuss the main result, homomorphisms to Hmod(S)
   showing that there are finitely many S at most 10 Hmod(S),
   each with a non-trivial graph of groups decoupling
   such homomorphisms.

Once the materials on check list above are gathered, please scan ALL materials and send to the
Computing Department. Return this form to Larry Patlage, Head of Computing (rm 214)

For Video Tapings - MSRI 9/2006
JIM CANNON: ‘Curvature has profound implications for the structure of groups and for algorithmic properties.’

BENSON: ‘Actually, his paper (in his book) is also amazing, and the motivation for the work I’m talking about today came partly from reading his paper.’

The title of the talk is:

‘Homomorphisms to the mapping class group’

The theorem which I hope to state towards the end of the talk is

**Theorem A.** Let \( \text{Mod}_S \) be the mapping class group of a (topologically finite) orientable surface \( S \) and let \( \mathcal{C} = \mathcal{C}(S) \) be the curve complex associated to \( S \). Let \( \Gamma \) be a finitely presented group. There is a finite collection of subgroups \( \mathcal{F}_{\Gamma,S} \) of \( \text{Mod}_S \) satisfying the following:

If \( \rho : \Gamma \to \text{Mod}_S \) is any homomorphism so that \( \rho(\Gamma) \) is not conjugate to an element of \( \mathcal{F}_{\Gamma,S} \) then there is a finite index subgroup \( \Gamma_0 \leq \Gamma \) and a nontrivial graph of groups \( \mathcal{G} \) so that \( \rho_0 = \rho|_{\Gamma_0} \) factors as \( \rho_0 = \phi \circ \psi \) where \( \psi : \Gamma \to \pi_1(\mathcal{G}) \) is surjective and \( \phi : \pi(\mathcal{G}) \to \text{Mod}_S \) is so that the images in \( \text{Mod}_S \) of the edge groups of \( \mathcal{G} \) each fix a curve on \( S \) (up to isotopy).

This is what I’ve been thinking about recently, and so what I want to talk about (I’d also like to get it on record that I’ve proved this theorem).

**EMPHASISE THAT** understanding sets of homomorphisms between groups is key to many topics in geometric group theory. (Acknowledge my debt to Zlil.) (Of course, before Zlil there was Morgan-Shalen and Culler, and Paulin, Bestvina, and of course Rips). Talk about other things I know: Isomorphism problem (and other algorithmic things), Hopf/co-Hopf property, MR diagrams, elementary theory... Particularly equations — ‘algebraic geometry over groups’.

but today we’re concentrating on a topological problem.
(NOTE: In the general context, post-conjugating the homomorphism is the same as changing the basepoint for the action – so it is natural to consider conjugacy classes of homomorphisms.)

Outline of talk:
• Apologise for talking about my latest theorem in an introductory workshop. However, my justifications for this are:

(1) It’s all I’ve been thinking about and I’m obsessed;
(2) I’m not going to give any of the details of the proof, except in the simplest case, where I’ll give a proof which is mostly unrelated to the one in the general case;
(3) I’ll try to bring out some themes from other things happening this week and last: Benson’s talks; Lisa’s talks from Womens; Cannon’s talks (negative curvature); complex of curves was important in the talks by Jeff, Ken and Yair in Kleinian groups workshop...
(4) I’ll also bring out what I think is a different important theme in GGT, namely ...

• Understanding sets of the form $\text{Hom}(\Gamma, G)$. Give reasons for studying $\text{Hom}(G, H)$: (see above, and also bundles).

(1) $H = \mathbb{R}$ – first cohomology;
(2) $H = \text{SL}(2, \mathbb{C})$ – representation varieties;
(3) But we’re going to concentrate on $H$ being discrete (fix $H$ and let $G$ be arbitrary):
   (a) Subgroup structure of $H$;
   (b) If $G = H$ then we study $\text{Aut}(H)$, or whether $H$ is Hopfian, or co-Hopfian;
   (c) ‘algebraic geometry over $H$’;
   (d) The universal theory of $H$, leading to logic over groups...
   (e) For us today, we’re interested in bundles...

• Surface bundles. Define. Motivate. Classify by homotopy classes of maps, which are the same as conjugacy classes of homomorphisms. Get to $\text{Mod}_S$ from $\text{Diff}^+(S)$ because the connected component of the identity in $\text{Diff}^+(S)$ is contractible. Therefore, of fundamental interest is understanding conjugacy classes of homomorphisms from an arbitrary group to $\text{Mod}_S$. (We might not go through this whole theory in the lecture, just mention that understanding bundles relates to understanding

In this section we should also mention characteristic classes and why cohomology may be less useful for understanding surface bundles than it is for vector bundles.

NOTE: I really want to understand conjugacy classes of homomorphisms, but at the moment all I have is results about homomorphic images.
• A brief detour to understand why negative curvature (in the target) helps understand homomorphic images. Explain the theorem in case the target is free (rather than \(\text{Mod}_5\)). This is easy (particularly since all subgroups of a free group is free), but it might be worth explaining how to put a lamination on the (triangular) presentation 2-complex of \(\Gamma\) and thereby get a splitting. THINK CAREFULLY ABOUT WHAT TO SAY HERE, because I'm not going to explain the proof of Theorem A at all. Say anything about hyperbolic groups? Limits, canonical representatives, etc.

*** Actually, let's prove the theorem in case of \(\text{SL}(2, \mathbb{Z})\) – pass to a finite index free subgroup and then prove it for a free group!

• Where does the negative curvature in \(\text{Mod}_5\) come from? The curve complex. Talk a little about the work of Masur and Minsky... (or just say 'The negative curvature in the curve complex (Masur-Minsky) can be used to study the mapping class group. This is the approach that we take, and we rely heavily on the results of Masur and Minsky'.)

• State the main theorem and the corollaries about all pseudo-Anosov subgroups of \(\text{Mod}_5\) and about Property (T) groups. (Mention Bowditch for all pseudo-Anosov surface groups.)

SOMEBWHERE: List the different kinds of elements of the mapping class group - talk about curve stabilisers (mapping class groups of surfaces of smaller 'complexity').

SOMEBWHERE: Talk about graphs of groups and how the hope is to split the group \(\Gamma\) up into simpler pieces – we understand the edge groups by induction? And hopefully the vertex groups are 'simpler'? NOTE that much of this is speculative in that there is no JSJ theory yet, or MR diagrams, or a real shortening argument, or a real sense of how to do this...

AS IN BENSON'S PAPER (In his book), we have bijections

\[
\begin{align*}
\text{Isomorphism classes of } S\text{-bundles over } B &\longleftrightarrow \text{Homotopy classes of maps } B \to B\text{Mod}_5 \\
&\longleftrightarrow \text{Conjugacy classes of representations } \rho : \pi_1 B \to \text{Mod}_5
\end{align*}
\]

(I'm taking into account torsion in \(\text{Mod}_5\) so I'm using \(B\text{Mod}_5\) rather than the moduli space.)

NOTE: \(B\text{Mod}_5\) is the classifying space for the group \(\text{Mod}_5\). Since the connected component of the identity in \(\text{Diff}^+(S)\) is contractible (Earle-Eells), and \(\text{Mod}_5 = \pi_0(\text{Diff}^+(S))\), there is a homotopy equivalence between the classifying spaces:

\[
B\text{Diff}^+(S) \simeq B\text{Mod}_5.
\]
Motivation: To understand $\text{Hom}(G, H)$.

For $\text{ex}$, $H = \mathbb{R} \rightarrow H^+$

$H = \text{SL}(2, \mathbb{C}) \rightarrow$ representation varieties.

$H$ discrete. If $G$ acts on $X$ and $\rho: G \rightarrow H$ homomorph $\Rightarrow G$ acts on $X$.

$G = H$ then $\text{Hom}(G, H) = \text{Aut} G$.

Is $G$ Hopfian? $G$ is Hopfian if every surjective homomorphism is injective.

Is $G$ co-Hopfian? (every injective homomorphism is surjective)

More generally, focusing on injective homomorphisms could lead to an understanding of $\text{sgn}$ of $H$.

Equations over $H$: Variables correspond to group elements. If we have variables $x_1, \ldots, x_n$, and want to understand all sols. to a system of

$$\sum_{i=1}^k w_i(x_1, \ldots, x_n) = 0,$$


A solution is a tuple $h \in H^n$ s.t. $w_i(h) = 0$.

Observe: $\text{Hom}(\langle x_1, \ldots, x_n \rangle, H) \uparrow 1 \downarrow$ solutions to $\{w_i(x)\}$ leads to study of logic over groups.
Example: Take $\Gamma = \Gamma(S, \delta = 1)$ be the $S$. Then, $\Gamma$ is a group of $\text{Mod}(S)$.

There are finitely many subgroups $\Gamma_i$ of $\text{Mod}(S)$ (which depend on $\Gamma$ and $\delta$). For $\Gamma_i \rightarrow \text{Mod}(S)$, $\delta$ is an oriented surface.

Theorem: $S$ compact oriented surface, $\Gamma$ big group.

When $S$ compact oriented surface, then we have a 1-1 correspondence by meridians of $S$.
\[ \langle h(a), h(b), h(c), h(d) \rangle \leq \text{Mod}(S) \text{ is the image of a surface group in } \text{Mod}(S). \]

Now for any \( k \), \( \langle h(a), h(b), \overline{T}^k h(c) \overline{T}^{-k} h(d) \overline{T}^k \rangle \) is a (hopefully different) image of \( T h(S) \) in \( \text{Mod}(S) \).

\[ \overline{g} \in \text{Mod}(S) \text{ is pseudo-} \text{Anosov if there is no pair } \langle f, k \rangle, k \in \mathbb{Z}, \text{ s.t. } g^k(f) = f. \]

There are free groups in \( \text{Mod}(S) \) all of whose non-identity elements are pseudo-\text{Anosov}. "All pseudo-\text{A}"

However, there are no known examples of one-ended all pseudo-\text{Anosov} subgroups.

**Corollary:** If \( \Gamma \) is one-ended, then there are at most finitely many conjugacy classes of all pseudo-\text{Anosov} subgroups of \( \text{Mod}(S) \) which are isomorphic to \( \Gamma \).

(In particular, \( \Gamma \) a surface group previously proved by Bowditch.)

**Proof:** Suppose infinitely many. There is an injective homomorphism \( \phi : \Gamma \to \text{Mod}(S) \) (all pseudo-\text{Anosov}) lying in case (iii).

Then \( \exists \Gamma_0 \leq \Gamma \) which admits a nontrivial graph of groups decomposition with edge groups mapping into curve stabilizers.

Thus \( \Gamma_0 \) splits over \( 3 \) \( \Rightarrow \) \( \Gamma_0 \) has infinitely many ends.

But then \( \Gamma \) has infinitely many ends. \( \text{Yield.} \)
Corollary: Let $P$ be a f.g. group with property (T). Then there are at most finitely many conjugacy classes of monomorphisms of $P$ in $\text{Mod}(S)$.

One case of the proof of the theorem:

Case: $S = T^2$, $\text{Mod} S = \text{SL}(2, \mathbb{Z})$. Take $P \to \text{SL}(2, \mathbb{Z})$.

$\Gamma_0 \to \text{free grp.}$