Note Taker Checklist Form - MSRI

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Talk Title and Workshop assigned to:

Kleiner's proof of the polynomial growth theorem
Topics in geometric group theory

Lecturer (Full name): David Fisher
Date & Time of Event: 11/9/07 2:00 pm

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do NOT use pencil or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords:

2. Please summarize the lecture in 5 or less sentences.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patagie, Head of Computing (rm 214)

For Video Tapings - MSRI 9/2006
Thm (Gromov) If a finitely generated group $G$ has polynomial growth, then $G$ has a finite index nilpotent subgroup.

Plan: 
1. Reductions to a statement about harmonic analysis
2. Harmonic functions/maps
3. Two key inequalities
4. Prove the theorem

Gromov's strategy: Induction, enough to show $\exists G \to \mathbb{Z}$.

(Wo Artin-Wolf: polycyclic group with poly. growth has finite index nilp. subgroup)

Enough to show $\exists$ image linear rep of $G$.

$\Rightarrow G \to \mathbb{Z}$ using Tits-Muller-Wolf's (similar) $\Rightarrow$ amenable linear reps are virtually solvable.

Theorem (Mok, Korevaar-Shoen) If a group $G$ doesn't have property (T), then there is an isometric action of $G$ on a Hilbert space.

And $\exists$ unbounded $G$-equivariant harmonic map $f : \text{Cay} (G, S) \to H$.

We'll use $G = \text{Cay} (G, S)$.

Observation: It is enough to see that $f(G)$ is contained in a fin. dim' subspace. This gives a map $\rho : G \to \text{Aut} \\mathbb{R}^n$ with $|\rho (G)| = \infty$.

Fact 1: If $G$ has poly growth, then it does not have property (T).

To show $f(G)$ is contained in a fin. dim' subspace, suffices to show that the space of Lipschitz harmonic functions on $G$ is fin. dim.
Why? \( \forall v \in H \quad L(v) : H \to \mathbb{R} \quad L(v) = \langle v, v \rangle \)

then \( L : \mathcal{F} \to \mathbb{R} \) is Lipschitz & harmonic.

**Theorem (Klein)** The space of harmonic functions of polynomial growth on a group \( G \) of polynomial growth is finite dimensional.

**Harmonic maps/functions:** \( G = \text{Cay}(G, S) = (V, E) \quad \mathcal{F} : V \to \mathbb{R} \) functions.

**Def:** \( \mathcal{F} : V \to \mathbb{R} \) is harmonic if \( \forall v \in V \quad g(v) = \frac{1}{15(v)} \sum_{d(v, w) = 1} g(w) \)

value at \( v \) is average of values of nearest neighbors.

\( \Rightarrow \quad \sum_{d(v, w) = 1} (g(v) - g(w)) = 0 \)

Can define \( f : V \to \mathbb{H} \) harmonic in the same way:

\( f : V \to \mathbb{H} \quad X = f(e) \quad (G, \text{complex graph on arc neighbors} \ f(e)) \)

\( E(f) = \sum_{s \in S} d^2(s, X) \)

**Def:** \( f \) is harmonic if it minimizes \( E \) among equivariant maps.

Exercise: Show defs. of \( f \) harmonic are equivalent.

- Compute \( \text{DE} \) min. \( \text{where DE} = G \quad E \) is convex.

Using this def., it is clear that \( L \mathcal{F} \) is harmonic.
Two inequalities \( q : V \to \mathbb{R} \), \( \forall g \in E \to \mathbb{R} \):

\[
\forall g \in E \to \mathbb{R} : \quad \|g(e) - g(\delta(e))\|_{\delta(e)} \leq e_{\delta(e)}.
\]

Poincaré inequality \( f : B(3R) \to \mathbb{R}, \ B(3R) : \) some ball of radius \(3R\):

\[
\int_{B(R)} |f - f_{B(R)}|^2 \leq C R^2 \frac{V(2R)}{V(R)} \int_{B(3R)} |f| \quad \text{for } R \to \infty.
\]

Decaying: \( \frac{V(2R)}{V(R)} \leq D \) that doesn't depend on \( R \).

The above is true on any Cayley graph.

Reverse Poincaré inequality \( f : B(3R) \to \mathbb{R} \) harmonic:

\[
\text{then } \quad R^2 \int_{B(R)} |f|^2 \leq C \int_{B(3R)} |f|^2.
\]

Will give a proof, judicious these inequalities so the domains of integration are the same on both sides.

\((B(R) \text{ instead of } B(3R), B(3R) \text{ - wildly false but can 'go back and fix it' - this is conceptually much easier.)}\)
$V$ a fd space of Lipschitz harmonic functions on $G$.

$\dim V = k$.

$Q_R(u,u) = \sum_{B \in R} u^2$ quadratic form on $V$.

$Q_R$ is non-decreasing in $R$.

$$\lim_{R \to \infty} \frac{V(R) \det (Q_R)^{1/k}}{R^d} < \infty$$ for some $d$.

Choosing scales (all scales large enough $s$, $Q_R > 0$.

Given any $w \in N$, can choose $R_1$, $R_2$ s.t.

$$\left( \frac{R_2}{R_1} \right)^2 \leq \frac{1}{e^w}$$ and so the volume growth, both

near $R_1$ and $R_2$ and between $R_1$ and $R_2$, "looks doubling".

$$\frac{V(R_2)}{V(R_1)} < C(k)$$

Main application of choice of scales is to show if a cover of $B = \{B_i^j(k)\}$

of $B(R_e)$ s.t. 1) bounded # of overlaps

2) $|\Omega| \leq C e^w = 5$

("useful Vitali covering lemmas")

$\Phi : V \to IR^J, \Phi_j(u) = \frac{1}{|B_j(k)|} \sum_{B_j} u$

Goal: show $\Phi$ is injective. (shows $k < J$, finite)

Lemma: $\Phi$ injective, then $A_{R_2}(u,u) \leq CV(R_1) \left| \frac{\partial u}{\partial \nu} \right|^2 + CR_2^2 \left| \frac{\partial u}{\partial \nu} \right|^2$

Pt: $\Phi$ good cover + Poincare' inequality.
Given the lemma, apply reverse Poincaré inequality:

\[ Q_{R_1}(u,u) \leq CV(R_1) |\bar{\Phi}(u)|^2 + C \left( \frac{R_1}{R_2} \right)^2 \int_{\bar{B}(R_2)} u^2 \]

\[ Q_{R_2}(u,u) \leq CV(R_1) |\bar{\Phi}(u)|^2 + C \left( \frac{R_1}{R_2} \right)^2 Q_{R_2}(u,u) \cdot \]

- On \( \ker \bar{\Phi} \), depends only on # generators:

\[ Q_{R_2}(u,u) \leq C \left( \frac{R_1}{R_2} \right)^2 Q_{R_2}(u,u) \]

if \( C \left( \frac{R_1}{R_2} \right)^2 < 1 \), then (which we can choose values to get)

then \( u = 0 \).