Reading. §6.6–6.7.

Problems from the book:

- 3.6.2, 3.6.8
- Page 389 #3.19
- Page 669 #6.10, 6.12, 6.13

Additional problems:

1. A matroid is a combinatorial gadget that have applications in many fields of mathematics and beyond. There are many equivalent definitions, though proving equivalence may be difficult. Here are two definitions. Prove that they are equivalent whenever $\Delta$ is a finite simplicial complex.

   **Definition 1.** A matroid is a simplicial complex $\Delta$ that satisfies the exchange condition: for any two simplices $\sigma, \tau \in \Delta$ satisfying $|\sigma| < |\tau|$, there is a vertex $x \in \tau \setminus \sigma$ such that $\sigma \cup \{x\} \in \Delta$.

   We say that a simplicial complex is pure if the maximal simplices all contain the same number of vertices. Given a simplicial complex $\Delta$ and a subset $W \subseteq V$ of the vertex set, the vertex induced subcomplex on $W$ is the simplicial complex $\Delta(W) = \{\sigma \in \Delta \mid V(\sigma) \subseteq W\}$, where $V(\sigma)$ denotes the vertex set of the simplex $\sigma$.

   **Definition 2.** A pure simplicial complex $\Delta$ is a matroid if the vertex induced subcomplex $\Delta(W)$ is pure for every subset $W \subseteq V$ of the vertex set.

2. a. Let $U$ be a vector space over $\mathbb{R}$, and let $\Delta(U)$ be the set of sets of linearly independent vectors in $U$. For example, if $U = \mathbb{R}^n$, then any subset of the standard basis vectors is an element of $\Delta(U)$. Prove that $\Delta(U)$ is a simplicial complex.

   b. Prove that $\Delta(U)$ is a matroid, using one of the above two definitions.