When you hand in this problem set, please indicate on the top of the front page how much time it took you to complete.

**Reading.** §4.8–4.10.

**Problems from the book:**
- 4.8.4, 4.8.8, 4.8.10, 4.8.16
- 4.9.1

**Additional problems:**

1. Let \( X = \{1, 2, 3, 4\} \). Write out all of the even permutations of \( X \). Can you find a subset of four even permutations that is closed under composition?

   Is the composition of two even permutations even? Is the composition of two odd permutations odd?

2. A **group** is a non-empty set \( G \) together with a binary operation \( \cdot \) on \( G \) such that
   - (Closure) For each \( g, h \in G \), \( g \cdot h \in G \);
   - (Associativity) For each \( g, h, k \in G \), \( (g \cdot h) \cdot k = g \cdot (h \cdot k) \);
   - (Identity) There exists an element \( e \in G \) such that \( e \cdot g = g \cdot e = g \) for every \( g \in G \); and
   - (Inverses) For each \( g \in G \) there is a \( h \in G \) so that \( g \cdot h = h \cdot g = e \). We write \( h = g^{-1} \).

   If \( G \) is a group, and \( H \subset G \) is a also group (with the same operation \( \cdot \)), we call \( H \) a normal subgroup of \( G \). If for every \( g \in G \) and for every \( h \in H \), \( ghg^{-1} \in H \), we call \( H \) a normal subgroup.

   a. Show that \( S_n \) is a group. (This is called the symmetric group on \( n \) letters.) For any set \( X \), let \( \text{Sym}(X) \) denote the permutations on \( X \). Show that \( \text{Sym}(X) \) is a group, with operation equal to function composition.

   b. Let \( A_n \) denote the subset of \( S_n \) of even permutations. Show that \( A_n \) is a normal subgroup of \( S_n \). (It is called the alternating subgroup.) What is the cardinality of \( A_n \)? For any \( \sigma_1, \sigma_2 \in S_n \), show that \( \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \in A_n \).

   c. Show that \( A_4 \) contains a normal subgroup \( K_4 \) of cardinality 4, and that \( K_4 \) contains a normal subgroup \( D \) of cardinality 2. Thus, we have a chain of subgroups

   \[
   \{e\} \subset D \subset K_4 \subset A_4 \subset S_4
   \]

   such that each group is normal in the one immediately containing it, and the ratio of the cardinalities of any two successive groups is a prime number.

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\(^1\)OK, this problem is actually very annoying. Grapple with it a little bit, and then explain (very explicitly) how you would do the problem if you had infinite time.
d. The above does not hold for $S_n$, where $n \geq 5$. In particular, there is no chain of subgroups

$$\{e\} \subset G_n \subset \cdots \subset G_2 \subset G_1 \subset S_5$$

such that each subgroup is normal in the one immediately containing it, and the ratio of the cardinalities of any two successive groups is prime. This failure is related to the fact that we cannot solve a general 5th degree (and higher) equation by radicals. Give an example of a subgroup of $A_5$ that is not normal.

e. The order of an element in a finite group is the least positive number $n$ such that the $n$-fold product $g^n = g \cdots g = e$. (In a finite group, the order of an element is always a finite number. In an infinite group, if there is no such $n$, we say that $g$ has infinite order.) Determine the largest possible order of an element of $S^4$. Answer the same question for $S_5$, $S_6$, $S_7$, $S_8$, and $S_9$.

3. Let $A$ be an $n \times n$ matrix. The characteristic polynomial of $A$ is $\chi_A(t) = \det(tI-A)$.

a. Show that if $A$ and $B$ are $n \times n$ matrices satisfying $A = PBP^{-1}$, then $\chi_A = \chi_B$.

b. Is the converse true? If $A$ and $B$ have the same characteristic polynomial, is there an invertible matrix $P$ so that $A = PBP^{-1}$? Prove this, or find a counterexample.