Time limit: 165.75 hours

Your name:

- All answers must be written in complete English sentences. Please show your work and give explanations in complete sentences. Even if your final answer is incorrect, you may get partial credit if you give a coherent explanation your partial progress. Also, if you can’t answer a general question, try to pick some specific examples and say what you can. Coming up with partial answers is what mathematicians do for a living!
- You may consult any books or your notes, but in the interest of academic integrity, please give references to all sources that you make use of.
- The ONLY people you may speak to about the exam are Tara Holm and Juan Alonso. Unless something surprising happens, we will only answer questions of clarification.
- The exam has SEVEN problems. Please carefully write all your final answers, in order they are posed, add this page on the front, and staple all your pages together.

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**Academic integrity.** As always, you are expected to abide by the Cornell Code of Academic Integrity. This states, “A Cornell student’s submission of work for academic credit indicates that the work is the student’s own. All outside assistance should be acknowledged, and the student’s academic position truthfully reported at all times.”

By signing this, I agree to abide by the above guidelines and by the Cornell Code of Academic Integrity.

Your signature: 

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Good luck!!
1. a. Use regular old calculus to evaluate the improper integral \( \int_0^1 \ln(x) \, dx \). (Be careful: this is an improper integral. Please justify your steps!)

b. Now compute the \( n^{th} \) Riemann sum by breaking \([0, 1]\) into \( n \) equal pieces and using the right-hand endpoints. (No dyadic pavings, please!)

c. Prove \( \lim_{n \to \infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e} \).

(The above limit gives the coarse approximation \( n! \approx n^n e^{-n} \). Stirling’s approximation is better (and related!).)

2. Compute \( \int_R x \, |dx\, dy\, dz| \) and \( \int_R z \, |dx\, dy\, dz| \) over the region \( R \) that is the part of the cone \( 0 \leq z \leq h (1 - \sqrt{x^2 + y^2}) \) that lies in the first octant (\( x \geq 0, y \geq 0, z \geq 0 \)). Here, \( a \) and \( h \) are positive real numbers.

3. Let \( A \subseteq \mathbb{R}^n \) be a bounded subset, and let \( f : A \to \mathbb{R} \) be a non-negative function with \( \int_A f \, |d^n x| = 0 \). Show that \( \text{supp}(f) = \{ x \in A \mid f(x) \neq 0 \} \) has measure 0.

4. Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear transformation with characteristic polynomial \( \chi_T(t) \).

a. Suppose that \( \chi_T(t) \) has distinct roots \( \lambda_1, \ldots, \lambda_n \). Show that there is a basis \( v_1, \ldots, v_n \) of \( \mathbb{R}^n \) that satisfies \( T(v_j) = \lambda_j v_j \) for all \( j = 1, \ldots, n \).

b. Give an example of the failure of the above phenomenon if the eigenvalues of \( T \) are not distinct. How bad can the failure be? (This question is intentionally vague – part of answering it is interpreting it.)

5. Recall, from Problem Set 4, that a **group** is a non-empty set \( G \) together with a binary operation \( \cdot \) on \( G \) such that

- **(Closure)** for each \( g, h \in G \), \( g \cdot h \in G \);
- **(Associativity)** for each \( g, h, k \in G \), \( (g \cdot h) \cdot k = g \cdot (h \cdot k) \);
- **(Identity)** there exists an element \( e \in G \) such that \( e \cdot g = g \cdot e = g \) for every \( g \in G \); and
- **(Inverses)** for each \( g \in G \) there is a \( h \in G \) so that \( g \cdot h = h \cdot g = e \). We write \( h = g^{-1} \).

We say that \( G \) is **commutative** or **abelian** if \( g \cdot h = h \cdot g \) for all \( g, h \in G \). Let \( g^2 \) denote \( g \cdot g \).

You may use the fact that the inverse of a group element is unique.

a. Let \( G \) be a group. Show that \( G \) is abelian if and only if \( (g \cdot h)^{-1} = g^{-1} \cdot h^{-1} \) for all \( g, h \in G \).

b. Let \( G \) be a group. Show that if \( g^2 = e \) for all \( g \in G \), then \( G \) is abelian.

c. Let \( G \) be a finite group with an even number of elements. That is, the set \( G \) is finite, and has \( 2n \) elements for some \( n \in \mathbb{N} \). Show that there must be some element \( a \in G \) with \( a \neq e \) and satisfying \( a^2 = e \).
6. Recall that an $n \times n$ matrix is called **orthogonal** if $AA^T = I$, that $O(n)$ is the group of all orthogonal matrices, and that $SO(n)$ is the subgroup of $O(n)$ consisting of those orthogonal matrices with determinant 1. Consider $SO(n)$ as a subset of $\mathbb{R}^{n^2}$.

   a. What are the dimensions of $SO(1)$, $SO(2)$, $SO(3)$, and in general of $SO(n)$? Please prove your answer, or at least provide an argument that is straight-forward and could be turned into a proof at the expense of the clarity of the argument.

   b. Parametrise $SO(1)$, $SO(2)$, and $SO(3)$ and find their volumes (in the correct dimension).

7. a. Consider a $(2n + 1) \times (2n + 1)$ matrix $A = (a_{ij})$ such that $a_{ij} = 0$ if

   - $i < n + 1$ and $i < j < 2n + 1 - i$; or
   - $i > n + 1$ and $2n - i < j < i$.

   On which of the entries of $A$ does $\det(A)$ depend? How much effort would it take you to compute $\det(A)$ (in terms of $n$)?

   b. An $n \times n$ matrix $N$ is called **nilpotent** if for some $k$ (depending on $n$), $N^k = 0$. If $N$ is nilpotent, can you say anything about the eigenvalues of $N$? If $N$ is nilpotent, show that $I + N$ is invertible and find a formula for the inverse of $I + N$.

   c. Consider the $2n \times 2n$ matrix

   \[
   \begin{pmatrix}
   A & B \\
   C & D
   \end{pmatrix}
   \]

   where $A$, $B$, $C$, and $D$ are all $n \times n$ matrices.

   Suppose that $A$ is invertible. Under what additional conditions can you prove that

   \[
   \det \begin{pmatrix}
   A & B \\
   C & D
   \end{pmatrix} = \det(AD - CB)?
   \]

   If you relax your additional hypotheses, can you find a counter example?