Some follow-up from class:

1. If a linear transformation is invertible, it's inverse is automatically linear. So, I didn't need to make a big deal out of that in class! Can you prove this?

Where this kind of issue does come up is when you are studying continuous functions. There are continuous functions that, set-theoretically, have inverse functions, but the inverse functions need not be continuous. We will discuss some of the subtleties surrounding this issue later this semester.

2. I was completely wrong when I said:

$$\text{rank}(AB) = \text{rank}(BA).$$

Here's a counterexample:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$  

It is true that

$$\text{rank}(AB) \leq \text{rank}(A),$$

$$\text{rank}(AB) \leq \text{rank}(B),$$

and similarly for $BA$. Can you think about how you would prove this?
In any case, to show that an m×n matrix A, with m ≠ n, does NOT have both a left & right inverse, this is enough!

Suppose (without loss of generality) that m < n. Let B be the n×m candidate inverse. Then we want AB = I_{mxm} and BA = I_{n×n} to be identity matrices. BUT, we know

\[
\begin{align*}
\text{rank (A)} & \leq m \\
\text{rank (A)} & \leq n.
\end{align*}
\]

Think about why!

Thus, since m < n, \(\text{rank (A)} \leq m\). Similarly, \(\text{rank (B)} \leq m\). But then, by the inequalities, \(\text{rank (BA)} \leq m < n\).

Also, \(\text{rank (I_{n×n})} = n\).

Thus, we can't possibly have BA = I_{n×n}!

Think a little bit about the role that the rank-nullity theorem is playing here. If m ≠ n, then one of AB and BA must have a positive-dimensional kernel.
Note that for a linear transformation
\[ T: \mathbb{R}^n \rightarrow \mathbb{R}^n, \]
the dimension formula tells us that
\[ T \text{ is an isomorphism } \iff \ker(T) = \{0\} \]
\[ \iff \text{im}(T) = \mathbb{R}^n. \]
That is, for a linear transformation from \( \mathbb{R}^n \) to itself, surjectivity implies injectivity, and vice versa.

This is **NOT** true for infinite dimensional vector spaces! Consider
\[ T: \text{Pol}(\mathbb{R}) \rightarrow \text{Pol}(\mathbb{R}) \]
defined by
\[ T\left(\sum_{i \geq 0} c_i x^i\right) = \sum_{i \geq 0} c_{i+1} x^i. \]
Check that:
\begin{itemize}
  \item [a)] \( T \) is linear;
  \item [b)] \( T \) is surjective; and
  \item [c)] \( T \) is **NOT** injective.