Symmetry breaking and the Gross Prasad conjectures for orthogonal groups

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I would like to discuss some work on the Gross Prasad conjectures for tempered representations of real orthogonal groups and also examples extensions of these conjectures to Arthur-Vogan packets of orthogonal groups.

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**Gross Prasad conjectures concern**

\[ \dim \text{Hom}_H(U_G \otimes U_H, \mathbb{C}) \]

for tempered representations \( U_G \) and \( U_H \).
$\text{Hom}_H(U_G, U_H^\vee) = \text{Hom}_H(U_G \otimes U_H, \mathbb{C})$ is of interest for automorphic representations.
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**Periods of automorphic representations**

If \( \pi_A \) is an automorphic representation of \( G_A \) and \( f \in \pi_A \) then
\[ \int_{H_A} f(H_A) \, dh_A \] is defines a period of \( \pi_A \) if it converges and in particular an element in \( \text{Hom}_H(\pi_A, \mathbb{C}) \).

If \( \pi_A = \prod \pi_\nu \) then it defines an \( H \)-invariant linear functional on all representations \( \pi_\nu \) i.e an element in \( \text{Hom}_H(\pi_\nu, U_H) \) for a trivial representation \( U_H \).
\( \text{Hom}_H(U_G, U_H^\vee) = \text{Hom}_H(U_G \otimes U_H, \mathbb{C}) \) is of interest for automorphic representations.

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The integral is related to special values of \( L \)-functions and the linear invariant linear functionals are important in the relative trace formula of Jacquet.
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**Analysis on G/H**

We get a function on G/H by

\[
v \rightarrow T(U(g)v).
\]

This function is an eigenfunction of the invariant differential operators. Classical problem is determine the spectrum of the invariant differential operators on G/K. More generally determine the spectrum of invariant operators on G/H where H is fix points of an involution. Considered and solved by H. Schlichtkrull, E. van den Ban, T. Oshima and many others ...
For real orthogonal groups:
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Sun-Zhou and Aizenbud -Gurevich proved that for irreducible representations

\[ \text{Hom}_H(U_G, U_H) \text{ has at most dimension 1.} \]

B.Gross and D.Prasad have a precise conjecture for the dimension of \( \text{Hom}_H(U_G, U_H) \) if both \( U_G \) and \( U_H \) are tempered involving arithmetic invariants (epsilon factors)
A special case:

**DISCRETE Branching laws at the real place:** We consider irreducible unitary representations $U$ of $G$, whose restriction $U|_H$ to $H$ is a **direct sum** of irreducible representations of $H$ and determine explicitly the irreducible representations of $H$ in $U|_H$. 
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Complete description of all discrete branching laws for unitary representations with nontrivial cohomology for pairs $G=SO(n+1,1)$, $H=SO(n,1)$ which are usually not tempered is due to T. Kobayashi, Y. Oshima, but no arithmetic conjectures about them yet.
Symmetry breaking Integral operators and Differential operators for real groups

They are explicit operators in $\text{Hom}_H(U_G, U_H)$. For principal series representations this is an analysis problem related to the construction of intertwining operators.

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Symmetry breaking Integral operators and Differential operators for real groups
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So we proceed as follows:

We consider the space of $C^\infty$-vectors of $U_G$ and $U_H$. It is a Frechet space and $U_G$ and $U_H$ act continuously on this Frechet space and we obtain the smooth representations $U_G^\infty$ and $U_H^\infty$. 
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\[ \text{Hom}_H(U_G, U_H) \]

denotes the space of **continuous** (with respect to the Frechet topology) intertwining operators.
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We have

\[ \text{Hom}_H(U_G \otimes U_H, \mathbb{C}) = \text{Hom}_H(U_G, U_H) \]
Results about symmetry breaking operators:
Let $P = MAN$ and $P' = M' A' N'$ be Langlands decompositions of parabolic subgroups $G = SO(n + 1, 1)$ and $G' = SO(n, 1)$, respectively, satisfying

$$M' = M \cap G', \quad A' = A \cap G', \quad N' = N \cap G', \quad P' = P \cap G'.$$

Fix finite-dimensional representations
\[ \sigma : M \to GL_\mathbb{C}(V) \]
and
\[ \tau : M' \to GL_\mathbb{C}(W). \]
For \( \lambda \in \mathfrak{a}_C^* \) and \( \nu \in (\mathfrak{a}'_C)^* \), we define \( V_\lambda \) of \( P \) and \( W_\nu \) of \( P' \) by

\[
P = MAN \ni me^H n \mapsto \sigma(m)e^\lambda(H) \in GL_C(V)
\]

\[
P' = M'A'N' \ni m'e^{H'} n' \mapsto \tau(m')e^\nu(H') \in GL_C(W),
\]

Consider the induced representations

\[
I(P, V_\lambda) \text{ on the } C^\infty\text{-sections of } V_\lambda := G \times_P V_\lambda.
\]

\[
I(P', W_\nu) \text{ on the } C^\infty\text{-sections of } W_\nu := G \times_P W_\nu.
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For $\lambda \in a_C^*$ and $\nu \in (a_C')^*$, we define $V_\lambda$ of $P$ and $W_\nu$ of $P'$ by

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A example of a symmetry breaking operator in $\text{Hom}_H(I(P, V_\lambda)), I(P', (V_\lambda)|_{P'})$ is the restriction of a section $f$ in $I(P, V_\lambda)$ to the subvariety $G'/P'$. 

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Consider now the case $V = W$ trivial and the spherical principal series representations $I(\lambda)$ and $J(\nu)$ of $O(n+1,1)$ and $O(n,1)$ using normalized induction.
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In joint work with T. Kobayashi

$$\dim \text{Hom}_H(I(\lambda), J(\nu)) \leq 2$$
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\[ \dim \text{Hom}_H(I(\lambda), J(\nu)) \leq 2 \]

If \( I(\lambda) \) and \( I(\nu) \) are tempered then \( \dim \text{Hom}_H(I(\lambda), J(\nu)) = 1. \)
Remarks about the (symmetry breaking) intertwining operators

This $G'$ intertwining operator has similar form to the usual intertwining operator. Consider the spherical principal series representations as functions on the large Bruhat cell $N \sim \mathbb{R}^n$. 
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**Theorem 1. Joint with T. Kobayashi**

There exists an analytic family

$$\tilde{A}_{\lambda, \nu} \in \text{Hom}_{G'}(I(\lambda), J(\nu))$$

of symmetry breaking operators with the distribution kernel

$$\pi^* \frac{1}{L(1/2, e^{\lambda_1+\nu_1})L(1/2, e^{\lambda_1-\nu_1})} |x_n|^{\lambda+\nu-1/2} (|x|^2 + x_n^2)^{-\nu-(n-1)/2}.$$
Remark:

If the representations are tempered then the normalization factor is closely related to the

\[ L \text{-function of the representation } I(\lambda) \times J(\nu). \]
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If the representations are tempered then the normalization factor is closely related to the

\[ L(\lambda) \times J(\nu). \]

In this case \( \lambda \) and \( \nu \) are purely imaginary and \( I(\lambda) \times J(\nu) \) and \( I(-\lambda) \times J(-\nu) \) are isomorphic and the L-function of the representation \( I(\lambda) \times J(\nu) \) contains the factor

\[ L(1/2, e^{\lambda_1 + \nu_1})L(1/2, e^{\lambda_1 - \nu_1})L(1/2, e^{-\lambda_1 + \nu_1})L(1/2, e^{-\lambda_1 - \nu_1}) \]

The L-factor doesn’t have a pole and so the intertwining operator is nonzero for tempered representations.
Outline of ”a naive version” of the Gross Prasad conjectures for orthogonal groups

Idea
Consider the complex dual group $L(G)_{\mathbb{C}}$ of $G$. The tempered representations of orthogonal groups are parametrized by homomorphisms $\tau$ of the Weil group $W_{\mathbb{R}}$ into $L(G)_{\mathbb{C}}$. 
Outline of ”a naive version” of the Gross Prasad conjectures for orthogonal groups

Idea
Consider the complex dual group $L(G)_\mathbb{C}$ of $G$. The tempered representations of orthogonal groups are parametrized by homomorphisms $\tau$ of the Weil group $W_\mathbb{R}$ into $L(G)_\mathbb{C}$.

Unfortunately if we look at a homomorphism

$$\tau : W_\mathbb{R} \to L(G)_\mathbb{C}$$

we cannot easily read of the inner form for which this defines a representation. It defines packet $A_G(\tau)$ of representations of several forms of orthogonal groups.

Similarly we can define a packet $A_H(\tau_H)$.  

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We now to define a subset $\mathcal{A}$ of compatible representations of $\mathcal{A}_G(\tau) \times \mathcal{A}_H(\tau_H)$. 
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Gross Prasad Conjecture (simple version)
There exists exactly one pair $\pi_G \times \pi_H \in \mathcal{A}$ so that

$$\text{Hom}_H(\pi_G, \pi_H) = 1$$
We now define a subset $\mathcal{A}$ of compatible representations of $\mathcal{A}_G(\tau) \times \mathcal{A}_H(\tau_H)$.

Gross Prasad Conjecture (simple version)
There exists exactly one pair $\pi_G \times \pi_H \in \mathcal{A}$ so that

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Furthermore an $\epsilon$-factors determines the pair $\pi_G \times \pi_H$. 
A simple first example:

\[ G = SO(2, 1) = GL(2, \mathbb{R})/D^{+}, \quad H = SO(2). \]

Every representation \( U_H \) of \( H \) is tempered and are parametrized by the even integers. The tempered representations of \( G \) are spherical principal series representations and discrete series representations.
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If \( U_G \) is a spherical principal series representation and \( U_H \) is any representations of \( H \), then the set \( \mathcal{A} = \{ U_G \times U_H \} \) satisfies the G-P conjecture.
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If \( \pi \) is a discrete series representation and \( F_\pi \) the finite dimensional representation with the same infinitesimal character as \( \pi \), then for every representation \( U_H \) the set

\[ \mathcal{A} = \{ \pi \times U_H, F_\pi \times U_H \} \]

satisfies the G-P conjecture.
Problem: \( F_\pi \) is not tempered
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Solution: Consider tempered representations of $SO(2, 1) \times SO(2)$ and of $SO(3) \times SO(2)$ instead and consider the final dimensional representation $F_\pi$ as a representation of $SO(3)$. The set $\mathcal{F}$ now contains 2 irreducible tempered representations of $SO(2, 1) \times SO(2)$ and one representation of $SO(3) \times SO(2)$ and

$$\bigoplus_{U \in \mathcal{F}} \dim \text{Hom}_H(U, \mathbb{C}) = 1$$
Second Example

Assume that both $SO(p,q)$ and $SO(p-1,q)$ has discrete series representations

In this case $\mathcal{A} = \mathcal{A}_G(\tau) \times \mathcal{A}_H(\tau_H)$

Gross-Wallach proved about 20 years ago that under some additional assumption on the function there exists a pair $\pi_G, \pi_H \mathcal{A}$ so that

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\[
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Open Problem: There are no other such pairs in \( \mathcal{A} \).
I am interested here in the case of **tempered representations** which are not **discrete series representations** and also to see if there is an extension of this conjecture to cohomological relevant representations.
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To state the conjecture I have to introduce Vogan "packets" of representations of a collection of orthogonal groups. We need to discuss

- Inner forms of orthogonal groups
- Relevant inner forms (and relevant parabolic subgroups)
- L-groups and Vogan packets of representations
Inner forms:

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First case

$p+q = 2n+1$, Consider $G_0 = SO(n,n+1)$, i.e. that the form defining $G_0$ has signature $n,n+1$

Consider the family $SO(n-2r, n+1+2r)$ for an integer $r$ with $|r| \leq n$ Note this family contains the pure inner form $G_0$, a compact form is always in this family and either $SO(2n,1)$ or $S0(1,2n)$ is in this family.
Second case:

\( p+q = 2n \cong 0 \mod 4 \). We may take \( G_0 = \text{SO}(n,n) \), i.e. the form defining \( G_0 \) has signature \( n,n \). Consider the family \( S_0(n-2r, n+2r) \mid |r| \leq n \). Note the family contains \( G_0 \), the compact form is always in this family and that \( \text{SO}(2n-1,1) \) or \( S_0(1,2n-1) \) is not in this family.

Since we are interested in families which contain rank one orthogonal groups we can ignore this family.
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Since we are interested in families which contain rank one orthogonal groups we can ignore this family.

Instead we consider \( G_0 = \text{SO}(n-1,n+1) \) and the family \( \text{S}0(n-2-1r, n+2r+1) \mid r \mid \leq n \) which contains \( \text{SO}(2n-1,1) \) or \( \text{S}0(1,2n-1) \).
Third case:

\( p+q = 2n \not\equiv 0 \mod 4 \). We may take \( G_0 = \text{SO}(n,n) \). We consider the family \( \text{SO}(n-2r, n+2r) \mid r \mid \leq n \). Note the compact form is not in this family and either \( \text{SO}(2n-1,1) \) or \( \text{SO}(1,2n-1) \) is in this family.
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Relevant families of pairs of inner forms
Consider a family \( G \) of real forms \((G,H)\) of \( \text{SO}(n+1,\mathbb{C}) \times \text{SO}(n,\mathbb{C}) \) where both groups are in one of the families defined above.
**Third case:**

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**Relevant families of pairs of inner forms**

Consider a family \( \mathcal{G} \) of real forms \((G,H)\) of \( \text{SO}(n+1, \mathbb{C}) \times \text{SO}(n, \mathbb{C}) \) where both groups are in one of the families defined above.

We are interested in the situation \( H \subset G \). So we say that a pair in \( \mathcal{G} \) is relevant if this is satisfied. We denote this subset of \( \mathcal{G} \) by \( \mathcal{G}_r \).
Examples of relevant families of pairs of inner forms;

First example:
\[ G_r = \{ \text{SO}(n-2r, n+1+2r),\ H_r = \text{SO}(n-2r, n+2r) \} \]

Remark: If \( n \) is odd then this family does contain a pair of rank one orthogonal groups
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Second example:
\[ G_r = \{ S_0(n-2r, n+1+2r), \quad H_r = S_0(n-2r-1, n+2r +1) \} \]

Remark: \( n \) is even then this family does contain a pair of rank one orthogonal groups
Third example:
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Remark: If n is odd then this family contains a pair of rank one orthogonal subgroups.
Third example: 
\[ \mathcal{G}_r = \{ \text{SO}(n+2r, n-2r), \ H_r = \text{SO}(n-1+2r, n-2r) \} \]

Remark: If \( n \) is odd then this family contains a pair of rank one orthogonal subgroups.

A choice of compatible maximal parabolic subgroups \( P_G, P_H \) with \( P_H \subset P_G \) in the rank one pairs in \( \mathcal{G}_r \) defines pairs of relevant maximal parabolic subgroups for all members of a family of relevant inner forms.
An example:

Consider the case $SO(4,1)$, $SO(3,1)$
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Relevant parabolic subgroups have Levi subgroups
$(\text{SO}(3)_{\mathbb{R}^*}, \text{SO}(2,1)_{\mathbb{R}^*})$ and $(\text{SO}(1,2)_{\mathbb{R}^*}, \text{SO}(2)_{\mathbb{R}^*})$. 
The Langlands dual group of $\text{SO}(4,1)$ is $\text{Sp}(4, \mathbb{C})$
The Langlands dual group of $\text{SO}(3,1)$ is $\text{SO}(4, \mathbb{C})$
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The Langlands dual group of $SO(3,1)$ is $SO(4,\mathbb{C})$.

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The Weil group $W_{\mathbb{R}}$ is a non split extension of the Galois group $\{1, \tau\}$ by $\mathbb{C}^*$.

Consider 4 dimensional symplectic and orthogonal representations of $W_{\mathbb{R}}$ defining a tempered representations of $SO(4,1) \times SO(3,1)$ induced from relevant parabolic subgroups. These representations of $W_{\mathbb{R}}$ define also tempered representations of $SO(2,3) \times SO(1,3)$. 
The Langlands dual group of $SO(4,1)$ is $Sp(4, \mathbb{C})$.

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The Weil group $W_R$ is a non split extension of the Galois group $\{1, \tau\}$ by $\mathbb{C}^*$.

Consider 4 dimensional symplectic and orthogonal representations of $W_R$ defining a tempered representations of $SO(4,1) \times SO(3,1)$ induced from relevant parabolic subgroups. These representations of $W_R$ define also tempered representations of $SO(2,3) \times SO(1,3)$.

We obtain a set $\mathcal{A} = \{\tilde{G} \times \tilde{H}\}$ with 2 representations of groups $\tilde{G} \times \tilde{H}$.
Gross Prasad conjectures in this case:

If \( L(1/2, \tau_{sp} \times \tau_{orth}) \neq 0 \) exactly one of these representations has 

an \( \tilde{H} \)-invariant linear functional

**Theorem 2.** Consider a family \( \mathcal{G} \) which contains the pair \( G=SO(4,1), H=SO(3,1) \) and a family \( \mathcal{A} \) of representations of the groups in \( \mathcal{G} \) so that the tempered spherical principal series representations of \( G \times H \) are in \( \mathcal{A} \). This family satisfies the G-P conjecture
Theorem 3. Consider a family $\mathcal{G}$ which contains the pair $G = SO(5,1), H = SO(4,1)$ and a family $\mathcal{A}$ of representations of the groups in $\mathcal{G}$ so that the tempered spherical principal series representation on $G \times H$ are in $\mathcal{A}$. This family $\mathcal{A}$ satisfies the $G$-$P$ conjectures.
Theorem 3. Consider a family $\mathcal{G}$ which contains the pair $G = \text{SO}(5,1)$, $H = \text{SO}(4,1)$ and a family $\mathcal{A}$ of representations of the groups in $\mathcal{G}$ so that the tempered spherical principal series representation on $G \times H$ are in $\mathcal{A}$. This family $\mathcal{A}$ satisfies the $G$-$P$ conjectures.

A consequence of the work with T. Kobayashi

Theorem 4. Consider a family $\mathcal{G}$ which contains the pairs $G = \text{SO}(n+1,1)$, $H = \text{SO}(n,1)$ and a family $\mathcal{A}$ of the groups in $\mathcal{G}$ so that the tempered spherical principal series representations of $G \times H$ are in $\mathcal{A}$. This family contains at least one representation with an $\tilde{H}$ invariant linear functional.
Is is possible to generalize this idea to representations which are not tempered?
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Possible interesting families:
Find families of representations of cohomologically induced representations $A_q(\lambda)$ (generalization of Adams/Johnson packets)
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Families of representations which contain representations of arithmetic interest, i.e. reps in Arthur packets
Some experimental results:

Consider the complementary series representation $I(3)$ of $SO(5,1)$ which is isomorphic to a subrepresentation of $L^2(SO(5,1)/SO(4,1))$. The complementary series representation $I(2)$ of $SO(4,1)$ is isomorphic to a subrepresentation of $L^2(SO(4,1)/SO(3,1))$. For these orthogonal groups the Ramanujan conjecture doesn’t hold and so these representation are infinity components of automorphic representations.

The infinitesimal character of these representations is singular and there are only 2 irreducible unitary representations in the family $\mathcal{A}$ which contains $I(3) \times I(2)$. 
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Joint work with Kobayashi implies In this packet G-P is true.
We consider the unitary infinite dimensional subquotients $T_G$ and $T_H$ of the spherical principal series representations of $SO(5,1)$ respectively $SO(4,1)$ with infinitesimal character $\rho$. These representations are automorphic representations with nontrivial cohomology, are holomorphically induced from a $\theta$ stable parabolics with $L = T^1 SO(3,1)$ respectively $L = T^1 SO(2,1)$. 
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Joint with Kobayashi

$$\dim \text{Hom}_{SO(4,1)}(T_G, T_H) = 1$$

$T_G$ and $T_H$ are both members of an enlarged Arthur packet $A_{\text{coho}}$ of representations in which $G$-P holds.