Abstract. We outline some open problems to be discussed in a Simons Symposium discussion session.

Note: Feel free to skip the background sections entirely if you are happy to consider only families of
cusps (of GL(2) over \( \mathbb{Q} \)) of varying level and weight. The problems are already open in this case
to my knowledge, despite some partial results. From here on every cuspform is assumed to be an eigen-
vector for Hecke operators and normalized. Alternatively one can think of them as cuspidal automorphic
representations of GL(2, \( \mathcal{A} \)).

1. Background: Harmonic families

The reference for the definition of families is [SST14] (which has expanded from [Sar08]).

Let \( G \) be a connected reductive split group over \( \mathbb{Q} \). (This is for simplicity; one could remove the splitness
assumption and work over a finite extension of \( \mathbb{Q} \).) Let \( r : \hat{G} \to \text{GL}(n, \mathbb{C}) \) be a faithful representation of \( \hat{G} \).

Let \( \mathcal{H} \) be a set of cuspidal (or discrete) automorphic representations of \( G(\mathcal{A}) \) subject to local constraints at
finitely many places. (Technically speaking, we assume that the local constraint at each place is given by
a well-behaved subset of the unitary dual with positive Plancherel volume.)\(^1\) For instance one can assume
that at a fixed place, every member of \( \mathcal{H} \) is a particular discrete series, or belongs to a particular Bernstein
center, or unramified.) A harmonic family \( \mathfrak{F} \) is a collection of \( L \)-functions
\[
\mathfrak{F} = \{ L(s, \pi, r) : \pi \in \mathcal{H} \}
\]
(or the corresponding automorphic representations of \( \text{GL}(n, \mathcal{A}) \) assuming the Langlands functoriality) such
that the cardinality of \( \mathfrak{F} \) is infinite. Just to distinguish \( \mathcal{H} \) from \( \mathfrak{F} \), let’s name the former a pre-harmonic
family, which is a random terminology I just cooked up.

To fix the idea one could consider the case when \( G = \text{GL}(2) \) and \( r \) is the standard representation. By
imposing a local condition that the infinite component be discrete series, we may restrict ourselves to a
family of cuspforms of weight \( \geq 2 \), with possibly further conditions. As an example, one could fix weight and
push level \( N \) to infinity (while \( N \) remains prime to a particular integer if you like), or fix level and push
weight to infinity (while \( k \) could run over a subsequence, such as the set of positive even numbers, if you
like).

Another important type of families considered in [SST14] is a geometric family, which is loosely speaking
a family of \( L \)-functions (of the same size) arising from a family of algebraic varieties over \( \mathbb{Q} \). However such
a family is going to be disregarded in this note since it is not so interesting for Problem 1 below. We could
think of Problem 2, but still a harmonic family would be more amenable to available tools.

There are quite a few interesting arithmetic statistical questions about harmonic families (and geometric
families). I would like to focus on two questions, detailed below, in the discussion I am going to moderate.

2. Background: Algebraic automorphic representations and field of rationality

To set up the scene for the first question, let us recall some basic definitions in a casual manner. The two
main references are [Clo90] and [BG]. One can also see [ST, §2].

Let \( \pi \) be an automorphic representation of \( G(\mathcal{A}) \). The field of rationality of \( \pi \) is a subfield of \( \mathbb{C} \) defined as
\[
\mathbb{Q}(\pi) := \{ a \in \mathbb{C} : \sigma(a) = a, \ \forall \sigma \in \text{Aut}(\mathbb{C}) \text{ s.t. } \pi^\sigma \simeq \pi \},
\]

\(^{1}\)It is an interesting question how much we can relax the condition on the Plancherel volume \( > 0 \), cf. [SST14, 3.3].
where $\pi^\sigma$ is the twist of (the underlying $\mathbb{C}$-vector space for) $\pi$ by $\sigma$. When a cuspform $f$ gives rise to $\pi$ (with suitable normalization) we have $Q(\pi) = Q(f)$, where $Q(f)$ is the extension field of $Q$ generated by the coefficients in the $q$-expansion of $f$. Note that $Q(f)$ is also called the coefficient field or Hecke field of $f$.

A general automorphic representation is of transcendental nature so one wouldn’t expect $Q(\pi)$ to be finite over $Q$ in general. However an observation by Clozel, extended further by Buzzard and Gee, is that a simple condition at infinity should govern the algebraicity/transcendence of $\pi$. We say that $\pi$ is $C$-algebraic if the infinitesimal character of $\pi_{\infty}$ is “algebraic” after twisting by the half sum of all positive roots (or equivalently if the $L$-parameter of $\pi_{\infty}$ on $W_\mathbb{C} = \mathbb{C}^\times$ is an algebraic character after the same kind of half-sum twist).

**Conjecture 2.1** (Clozel, Buzzard-Gee). $\pi$ is $C$-algebraic $\iff [Q(\pi) : Q] < \infty$.

If $\pi$ appears in the cohomology of locally symmetric spaces or Shimura varieties associated to $G$ then it is basically known to be $C$-algebraic. This fact is used to establish $\Rightarrow$ for such a $\pi$. The converse is harder in general and seems to be a matter of transcendental number theory. The implication $\Leftarrow$ is known only when $G$ is a torus. For our discussion it would be harmless to restrict to $\pi$ whose infinite part is a discrete series, so that $Q(\pi)$ is known to be a number field.

3. Topic: Field of rationality in families

The main references are [Ser97, §6] and [ST].

Let $\mathfrak{F}$ be a harmonic family as above (or in fact, one could formulate the same question for $\mathfrak{H}$, since the notion of field of rationality makes sense either on $G$ or on $GL(n)$). For an integer $A \geq 1$, define $\mathfrak{F}_{\leq A}$ to be the subset of $\mathfrak{F}$ consisting of $\pi$ such that

$$[Q(\pi) : Q] \leq A.$$  

Write $\mathfrak{F}(x)$ and $\mathfrak{F}_{\leq A}(x)$ for the corresponding subsets with conductor $\leq x$. Then the basic problem is

**Problem 3.1.** Fix $A$. What is the asymptotic behavior of $|\mathfrak{F}_{\leq A}|/|\mathfrak{F}|$, i.e. what is the limiting behavior of $|\mathfrak{F}_{\leq A}(x)|/|\mathfrak{F}(x)|$ as $x \to \infty$? (If the limit is zero then do we have that $|\mathfrak{F}_{\leq A}(x)| = O(|\mathfrak{F}(x)|^{1-\delta})$ for some $\delta > 0$?)

Let us make the problem concrete in the case of cuspforms of $GL(2)$. Let $S_k(N)$ be the set of cuspforms with weight $k$ and level $N$ (either $\Gamma_0$, $\Gamma_1$ or $\Gamma$) in the above sense (so that it is not the usual $\mathbb{C}$-vector space of cuspforms but a basis thereof). Define $S^{\leq A}_k(N)$ in the obvious way. We’d like to consider two types of families:

1. (level aspect) $k$ is constant, $N$ tends to $\infty$,
2. (weight aspect) $N$ is constant, $k$ tends to $\infty$.

The problem is about the limit of $|S^{\leq A}_k(N)|/|S_k(N)|$ in the level or weight aspect. The level aspect turns out to be more accessible. When there exists a prime $p$ such that all $N_m$ are prime to $p$ (for $m \gg 0$) then Serre proved that the limit is zero. Let me recall a summary of his argument from introduction of [ST]. The key point is to show that

$$\left|\left\{a_p(f) : f \in S_k(N_m)^{\leq A}\right\}\right| < \infty, \quad (3.1)$$

where $a_p(f)$ is the $p$-th Fourier coefficient of $f$. This follows from the fact that $a_p(f)$ is an algebraic integer which is the sum of a Weil $p$-number of weight $k-1$ and its complex conjugate. The condition $[Q(f) : Q] \leq A$ implies that $[Q(a_p(f)) : Q] \leq A$, so such a Weil number is a root of a monic polynomial in $\mathbb{Z}[x]$ whose degree and coefficients are bounded only in terms of $p$, $m$, and $A$. Clearly there are only finitely many such polynomials, hence (3.1).

Under a milder hypothesis [ST] showed the same (and generalized to some higher rank groups). Still the level aspect is not completely settled even for $GL(2)$, and this is worth discussing.

**Problem 3.2.** [[Ser97, Question, p.89]] What is the limiting behavior of $|S^{\leq A}_k(N)|/|S_k(N)|$ in the level aspect ($k$ fixed, $N \to \infty$), where $N$ is arbitrary and not assumed to be prime to $p$?

Even more mysterious is the weight aspect. This case seems completely open, and it’s less transparent whether the limit should be zero or not. (In [Ser97, p.28], Serre says “J’ignore ce qui se passe lorsque l’on fixe $N$ et fait varier $k$.”) If weight varies, we no longer have the finiteness of Weil integers, a crucial ingredient in the level aspect sketched above. I don’t know of any prediction from numerical data and encourage an expert on the modular form database to help.
Problem 3.3. What is the limiting behavior of $|S_k^{≤A}(N)|/|S_k(N)|$ in the weight aspect ($N$ fixed, $k \to \infty$)? Can we develop some approach or heuristics, or at least collect numerical data to predict the answer?

Remark 3.4. When $N = 1$, Maeda’s conjecture asserts that the elements of $S_k(1)$ are Galois conjugate over $\mathbb{Q}$ for all $k$. In particular it would imply that $|S_k^{≤A}(1)|/|S_k(1)| = 0$ for all $k \gg 0$.

There are variants of the above question. For instance, one could replace the condition $[\mathbb{Q}(\pi) : \mathbb{Q}] ≤ A$ with the analogous degree over the maximal cyclotomic extension of $\mathbb{Q}$, or local extension degrees such as $[\mathbb{Q}_q(\pi) : \mathbb{Q}_q] ≤ A$ for a prime $q$. For the latter see the relevant question by Buzzard, cf. Questions 4.3 and 4.4 of [Buz05]. The former question asks, provided $N = 1$, whether $[\mathbb{Q}_2(\pi) : \mathbb{Q}_2] ≤ 2$ for all $\pi$ (as weight varies).

It would be intriguing to investigate analogous questions with $p$-adic families of automorphic forms in place of harmonic families. Some definite steps in this direction were taken by Hida [Hid11], [Hid]. Roughly speaking he showed that the field of rationality grows indefinitely in a Hida family (slope 0 family) of Hilbert modular forms (including elliptic modular forms) if the family is non-CM. His vertical (resp. horizontal) case is analogous to the weight (resp. level) aspect in this note. However his setting is quite different in that his field of rationality is, again roughly, the composite of the fields of rationality of all members (which are classical as opposed to genuinely $p$-adic) in a family. Nevertheless one can ask whether the $p$-adic approach sheds light on the above questions for harmonic families.

References


\footnote{Link available at http://www2.imperial.ac.uk/~buzzard/maths/research/papers/conjs.pdf. I thank Toby Gee for pointing me to Buzzard’s paper.}