MATH 2220: GUIDE TO INTEGRALS

There are many kinds of integrals in this course. This brief guide is supposed to help you to tell them apart.

1) Standard integral of a function \( f(x) \) of one variable on an interval \([a, b]\).

\[
\int_{a}^{b} f(x) \, dx.
\]

Represents area under the graph of \( f \) between \( x = a \) and \( x = b \).

2) Path integral of a scalar function \( f \) along a curve \( C \) with parametrization \( c(t) \), \( a \leq t \leq b \) in \( \mathbb{R}^3 \).

\[
\int_{C} f(x, y, z) \, ds = \int_{a}^{b} f(c(t)) \|c'(t)\| \, dt.
\]

Represents mass of a wire with shape \( C \) and density \( f(x, y, z) \).

Path integral of a scalar function \( f \) along a curve \( C \) with parametrization \( c(t) \), \( a \leq t \leq b \) in \( \mathbb{R}^2 \).

\[
\int_{C} f(x, y) \, ds = \int_{a}^{b} f(c(t)) \|c'(t)\| \, dt.
\]

Represents area of a curtain with base \( C \) and height \( f(x, y) \).

These two integrals don’t depend on the choice of parametrization of \( C \). Special case: \( f = 1 \) gives arc length.

3) Path (or line) integral of a vector field \( \mathbf{F} \) along a curve \( C \) with parametrization \( c(t) \), \( a \leq t \leq b \) in \( \mathbb{R}^3 \).

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{a}^{b} \mathbf{F}(c(t)) \cdot c'(t) \, dt.
\]

- Represents work done by \( \mathbf{F} \) on a particle moving along \( C \).
- Depends on orientation (direction) of \( C \), but not on the choice of parametrization.
- Other notations:

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot d\mathbf{s} = \int_{C} (F_1 dx + F_2 dy + F_3 dz).
\]
• If $C$ is a simple closed curve, this integral is sometimes written

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$  

(4) Double integral of a function $f(x, y)$ over a region $\Omega \subset \mathbb{R}^2$.

$$\iint_{\Omega} f(x, y)dA.$$  

Calculated by writing our region as $x$–simple or $y$–simple, also sometimes by conversion to polar coordinates or other change of variable. Special case: $f = 1$ gives area.

Change of variable formula: if $T : D \rightarrow T(D)$ is one-to-one then

$$\iint_{T(D)} f(x, y)dA = \iint_D f(T(u, v))|DT(u, v)|dudv.$$  

(5) Integral of a scalar function $f(x, y, z)$ over a parametrized surface $S$ with parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in D \subset \mathbb{R}^2$.

$$\iint_{S} f(x, y, z)dS = \iint_{D} f(\Phi(u, v))\|\Phi_u \times \Phi_v\|dudv.$$  

Represents mass of $S$ if $S$ has density $f(x, y, z)$ at a point $(x, y, z)$. Doesn’t depend on choice of orientation. Special case: $f = 1$ give surface area.

(6) Integral of a vector field $\mathbf{F}$ over a parametrized surface $S$ with parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in D \subset \mathbb{R}^2$.

$$\iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iint_{S} \mathbf{F} \cdot \mathbf{n}dS = \iint_{D} \mathbf{F}(\Phi(u, v)) \cdot (\Phi_u \times \Phi_v)dudv.$$  

where $\mathbf{n}$ is a unit normal in the direction $\Phi_u \times \Phi_v$.

• Depends on a choice of unit normal (orientation). Otherwise independent of the parametrization chosen.

• Represents the flux of the field $\mathbf{F}$ through $S$.

(7) Triple integral of a scalar function $f(x, y, z)$ over a region $B \subset \mathbb{R}^3$.

$$\iiint_B f(x, y, z)dV.$$  

Represents mass of a solid with shape $B$ and density $f$. Special case: $f = 1$ gives volume. Change of variable formula similar to the two-variable case. Two special changes of variables are cylindrical and spherical coordinates.
Cylindrical:
\[ \iiint_{B} f(x, y, z) dV = \iiint_{B_{cyl}} f(r \cos \theta, r \sin \theta, z) r dr d\theta dz. \]

Spherical:
\[ \iiint_{B} f(x, y, z) dV = \iiint_{B_{spher}} f(\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi. \]

(8) **Fundamental Theorem of calculus.** If \( F \) is an antiderivative of \( f \) then
\[ \int_{a}^{b} f(x) dx = F(b) - F(a). \]

(9) **Integral of a conservative field.** If \( f \) is a differentiable function and \( C \) is a curve with parametrization \( c(t), t \in [a, b] \), then
\[ \int_{C} \nabla f \cdot dr = f(c(b)) - f(c(a)). \]

(10) **Green’s Theorem.** If \( \Omega \) is any reasonable closed and bounded region in \( \mathbb{R}^2 \) and \( \partial \Omega \) is the boundary curve of \( \Omega \) with the “anticlockwise” orientation and \( \mathbf{F} = (P, Q) \) is a \( C^1 \) vector field on \( \Omega \) then
\[ \int_{\partial \Omega} \mathbf{F} \cdot dr = \iint_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA. \]

(11) **Stokes’ Theorem.** If \( S \) is an oriented surface in \( \mathbb{R}^3 \) and \( \partial S \) is the boundary of \( S \) (a collection of curves) with the induced orientation and \( \mathbf{F} \) is a \( C^1 \) vector field on \( \mathbb{R}^3 \) then
\[ \int_{\partial S} \mathbf{F} \cdot dr = \iint_{S} \text{curl}(\mathbf{F}) \cdot dS. \]

(12) **Divergence Theorem.** If \( B \) is a solid region in \( \mathbb{R}^3 \) and \( \partial B \) is the boundary surface of \( B \) and \( \mathbf{F} \) is a \( C^1 \) vector field on \( \mathbb{R}^3 \) then
\[ \iint_{\partial B} \mathbf{F} \cdot dS = \iiint_{B} \text{div}(\mathbf{F}) dV. \]

Note: the statements of these theorems have deliberately been left a little bit vague, but they apply in all the situations with which we are familiar.