(1) Section 1.1.3 Exercise 2b.

“The only even prime is 2.” There are many different ways of approaching the problem. One way is

\[ \forall n \in \mathbb{N} (n \text{ is even } \land n \text{ is prime } \implies n = 2). \]

The negation is

\[ \exists n \in \mathbb{N} (n \text{ is even } \land n \text{ is prime } \land n \neq 2). \]

That is, “There exists an even prime which is not equal to 2.”

(2) Section 1.1.3 Exercise 3b.

“Every nonzero rational number has a rational reciprocal.”

\[ \forall x \in \mathbb{Q} \setminus \{0\} \exists y \in \mathbb{Q} (xy = 1). \]

The corresponding statement with quantifiers reversed is:

\[ \exists y \in \mathbb{Q} \forall x \in \mathbb{Q} \setminus \{0\} (xy = 1). \]

This is false, because if \( y \in \mathbb{Q} \) is such that \( yx = 1 \) for all \( x \in \mathbb{Q} \setminus \{0\} \) then \( y = 2y = 1 \) which is impossible.

(3) Let \( A \) be a set and let \( P(a) \) be a statement about an element of \( a \). We write

\[ \exists! a \in A P(a) \]

for “there exists a unique \( a \in A \) such that \( P(a) \)”.

(a) Write the statement \( \exists! a \in A P(a) \) in a form which uses the quantifiers \( \forall \) and \( \exists \), and no connectives apart from \( \land, \lor \) and \( \neg \).

It can be written as

\[ \exists a \in A (P(a) \land \forall b \in A (\neg P(b) \lor b = a)). \]

(b) Write the negation of the statement from part (a).

\[ \forall a \in A (\neg P(a) \lor \exists b \in A (P(b) \land b \neq a)). \]
(4) Section 1.2.3 Exercise 2.

The set of all finite subsets of \( \mathbb{N} \) is countable.

**Proof:** Let \( A \) denote the set of all finite subsets of \( \mathbb{N} \). We need to define an injection \( f: A \to \mathbb{N} \). Let \( p_1 < p_2 < p_3 < \cdots \) denote the prime numbers listed in increasing order. Given a finite subset \( S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{N} \), relabel the \( x_i \) if necessary so that \( x_1 < x_2 \cdots < x_n \). Then define

\[
f(S) = p_1^{x_1}p_2^{x_2}\cdots p_n^{x_n}.
\]

By uniqueness of the decomposition of a natural number into a product of primes, if \( S \neq T \) then \( f(S) \neq f(T) \). Thus \( f \) is injective, as required.

(5) Section 1.2.3 Exercise 4.

**Proof:** Let \( A \) be an uncountable set. Let \( C \subset A \) be countable. Suppose for a contradiction that \( A \setminus C \) is countable. Then

\[
A = (A \setminus C) \cup C
\]

is a union of two countable sets. But, by a theorem from the lectures, a countable union of countable sets is countable. Thus, \( A \) is countable. This is a contradiction. Therefore, \( A \setminus C \) is uncountable.

(6) Section 1.2.3 Exercise 7. Let \( A \) be an infinite set. We wish to show that \( |\mathcal{P}(A)| > |A| \). First, we show that there is an injection \( A \to \mathcal{P}(A) \). This is clear, since we can map \( x \in A \) to \( \{x\} \in \mathcal{P}(A) \).

Next, we must show that there is no bijection \( A \to \mathcal{P}(A) \). Suppose \( f: A \to \mathcal{P}(A) \) is a bijection. Let

\[
Z = \{a \in A : a \notin f(a)\}.
\]

Since \( f \) is surjective, there exists \( b \in A \) with \( f(b) = Z \). Either \( b \in f(b) \) or \( b \notin f(b) \). If \( b \in f(b) \) then \( b \notin Z \) by definition of \( Z \). But this contradicts \( b \in f(b) = Z \). On the other hand, if \( b \notin f(b) \) then \( b \in Z \) by definition of \( Z \). But then \( b \in Z = f(b) \), a contradiction. Therefore, the bijection \( f \) does not exist.