MATH 2220 PRELIM (PRACTICE)

You have 1 hour 30 minutes to complete this exam. The exam starts at 7:30pm. Each question is worth 20 marks. There are 5 questions in total. The exam is printed on both sides of the paper.

Good luck!

(1) Calculate:
(a) \[ \int_0^1 \int_0^x \sin(y) \, dy \, dx. \]
(b) \[ \int_0^1 \int_0^1 \frac{3y^2}{x^r + 1} \, dx \, dy. \]

(2) Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[ f(x, y) = x(y + 1). \]
(a) Find all critical points of \( f(x, y) \) and determine their nature.
(b) Find the absolute maximum and minimum of \( f(x, y) \) on the disc \( x^2 + y^2 \leq 3 \).

(3) Let \( \Omega \) be the region enclosed by the plane \( 2x + 2y + z = 7 \) and the paraboloid \( z = x^2 + y^2 \).
(a) Sketch \( \Omega \).
(b) Set up a triple integral for the volume of \( \Omega \). Be sure to write all the limits of integration, but do not attempt to evaluate the integral.

(4) A rectangular cuboid has volume 100 cm\(^3\). Use Lagrange multipliers to find:
(a) The minimum possible surface area of the cuboid.
(b) The minimum possible sum of the lengths of the edges of the cuboid. [TURN OVER]
(5) Let
\[ f(x, y) = \frac{x}{\cos^2(y)} \]
(a) Find the second-order Taylor polynomial of \( f \) about \((0, 0)\). You may omit the remainder term.
(b) Show that the equation
\[ f(x, y) + f(x, z) + f(z, x) = 0 \]
uniquely determines \( z \) as a function of \( x \) and \( y \) near the point \((0,0,0)\).
(c) Find \( \frac{\partial z}{\partial x} \bigg|_{(0,0)} \).

[END OF PAPER.]

EXTRA QUESTIONS

Note: some of these are harder than what is likely to be on the exam. Also, see the textbook for more practice problems.

(1) Find the maximum possible volume of a circular cylinder with cross-sectional radius \( r \) and surface area \( S \).

(2) (Prelim 2 07) Let \( x > 1 \) and \( y > 1 \). Use Lagrange multipliers to find the maximum value of \( x^y \) subject to the constraint \( xy = 1 \).

(3) Solve the previous problem using one-variable calculus.

(4) Find the absolute maximum and minimum of \( f(x, y) = 3x+y \) on the disc \( x^2+y^2 \leq 1 \).

(5) Find numbers \( A_i \) such that \( 4x^2+3xy \) equals \( A_0 + A_1(x-1) + A_2(y-1) + A_3(x-1)^2 + A_4(x-1)(y-1) + A_5(y-1)^2 \).

(6) Use double integration to calculate the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \).

(7) Set up a triple integral for the volume enclosed by the parabolic cylinders \( z = x^2 \) and \( z = 1-y^2 \).

(8) Suppose \( y^5 + 2y + 3x = 0 \).
   (a) Use the implicit function theorem to show that this equation uniquely defines \( y \)
   as a function of \( x \) near \((-1,1)\).
   (b) Calculate \( \frac{dy}{dx} \bigg|_{x=-1} \).
(c) Find the second order Taylor polynomial of \( y \) near \((-1, 1)\).

(9) Show that \( f(x, y) = xy \) has only one critical point and that it is a saddle. Sketch the graph of \( f \). How is \( f \) related to the function \( g(x, y) = x^2 - y^2 \)?

(10) Find the absolute maximum and minimum of \( x \cos(y) \) on the rectangle \([0, 1] \times [0, 1]\).

(11) A very large tombstone is to be made in the shape of the box \([0, 1] \times [0, 5] \times [0, 10]\) in \( \mathbb{R}^3 \) (lengths are measured in feet). The tombstone is made of marble whose density at the point \((x, y, z)\) is given by \(100 - z^2\) kilograms per cubic foot. Set up a triple integral for the mass of the tombstone and compute it.

(12) Find the volume between the paraboloids \( z = 6 - x^2 - y^2 \) and \( z = x^2 + y^2 \) (you may need to use a table of integrals).

(13) Let \( a, b, D > 0 \). The hull of a ship is to consist of those points in \( \mathbb{R}^3 \) which lie above the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \) in the \( xy\)-plane and which lie below the plane \( \frac{D}{b} y + D = z \) and below the plane \( -\frac{D}{b} y + D = z \).

(a) Make a rough sketch of the hull.

(b) Calculate the volume of the hull (hint: it is easier to calculate half the volume. That is, the volume of the solid that lies above the region \( \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, y \leq 0 \) and below the plane \( \frac{D}{b} y + D = z \)).

(c) Given the constraint \( a + b + D = k \), find the values of \( a, b \) and \( D \) which maximize the volume of the hull.