MATH 2220 PRELIM 1

You have 1 hour 30 minutes to complete this exam. The exam starts at 7:30pm. Each question is worth 20 marks. There are 5 questions in total. You are free to use results from the lectures, but you should clearly state any theorems you use. The exam is printed on both sides of the paper. Good luck!

(1) State whether the following are true or false and justify your answer:

(a) The vectors $(5, 7, 1)$ and $(1, 0, -5)$ are orthogonal.
(b) The angle between the vectors $(0, 0, 1)$ and $(5, 7, -1)$ is $\pi$.
(c) If $L_1$ and $L_2$ are lines in $\mathbb{R}^3$ then there exists a plane containing $L_1$ and $L_2$.
(d) If $a$ and $b$ are vectors in $\mathbb{R}^3$ then $\|a \times b\| \leq \|a\|\|b\|$.
(e) If $a, b$ and $c$ are vectors in $\mathbb{R}^3$ then

$$\|(a - b) \times (c - b)\| = \|(b - a) \times (c - a)\|.$$

(2) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = \frac{z + 1}{e^x (\cos^2(y) + \frac{1}{2})}.$$

(a) State what it means for a function $f : \mathbb{R}^3 \to \mathbb{R}$ to be continuous.
(b) Show that $f$ is continuous. You may use the fact that the functions $e^x$ and $\cos(x)$ are continuous functions of one variable.
(c) Either find $\lim_{(x, y, z) \to (0, 0, 0)} f(x, y, z)$ or show that it does not exist.
(d) Calculate

$$\frac{\partial^5 f}{\partial z \partial z \partial x \partial y \partial y}
\big|_{(0,0,0)}.$$

(3) (a) Find the tangent plane $P_a$ to the surface $z = x^2 - y^3$ at $(1, 1, 0)$.
(b) Find the tangent plane $P_b$ to the surface $yz^2 = 2$ at $(0, 2, 1)$.
(c) Find the line of intersection of the planes $P_a$ and $P_b$. [TURN OVER]
(4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$ function with
\[
\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 5 \\
\left. \frac{\partial f}{\partial y} \right|_{(0,0)} = -3
\]
Define $g : \mathbb{R}^3 \to \mathbb{R}$ by
\[
g(u, v, w) = f(u + v + w, uvw).
\]
(a) Write down a function $h(u, v, w) : \mathbb{R}^3 \to \mathbb{R}^2$ such that $g = f \circ h$.
(b) Write down a formula for $\nabla g(u, v, w)$ in terms of the derivatives of $f$ and $h$.
(c) Find $\nabla g(0, 0, 0)$.
(d) Find the directional derivative of $g$ at the point $(0, 0, 0)$ in the direction of the vector $(3, 4, 0)$.

(5) The height $z$ of an island above sea level (in meters) is given by
\[
z = f(x, y) = \frac{2}{1 + x^2} + \frac{2}{1 + y^2}.
\]
(a) Draw the level curve $f(x, y) = k$ for the value $k = 2$.
(b) Calculate $\nabla f$.
(c) A pirate, Captain X, is marooned on the island. He is standing at the point $(-1, 1, 2)$ and wants to descend as quickly as possible. In which direction should he set off?
(d) Captain X has a wooden leg. He will stumble and fall if he has to move in a direction in which the rate of change of $f$ is more than $\frac{1}{2}$ or less than $-\frac{1}{2}$. In which directions could Captain X set off if he wants to descend as quickly as possible but without stumbling?