MATH 2220 HW4.

Due Wednesday 24 September

(1) Use the derivative to estimate the value of \( \cos(0.02 \cos(-0.03)) \).

Let \( f(x, y) = \cos(x \cos(y)) \). Let \( x_0 = (0, 0) \). Let \( x = (0.02, -0.03) \). We estimate \( f(x) \) by using the linear approximation:

\[
f(x_0) + \nabla f(x_0)(x - x_0)
\]

We have \( f_x = -\sin(x \cos(y)) \cos(y) \) and \( f_y = \sin(x \cos(y)) x \sin(y) \). Therefore \( f_x(0, 0) = 0 = f_y(0, 0) \) and \( f(0, 0) = 1 \). So

\[
f(x_0) + \nabla f(x_0)(x - x_0) = 1 + \begin{bmatrix} 0 & 0 \\ -0.03 & 0 \end{bmatrix} = 1.
\]

(2) Section 2.5 p.159-163

(a) \# 11.

Using the chain rule, first calculate \( D(f \circ T)(1, 0) \).

\[
D(f \circ T)(1, 0) = Df(T(1, 0)) DT(1, 0) = Df(1, \log \sqrt{2}) DT(1, 0).
\]

\[
Df(u, v) = (-\sin u \sin v, \cos u \cos v) \quad \text{and}
\]

\[
DT(s, t) = \begin{bmatrix} -t^2 \sin(t^2 s) & -2ts \sin(t^2 s) \\ \frac{2s}{\sqrt{1+s^2}} & 0 \end{bmatrix}
\]

So

\[
DT(1, 0) = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}
\]

Therefore,

\[
Df(1, \log \sqrt{2}) DT(1, 0) = (-\sin(1) \sin(\log \sqrt{2}), \cos(1) \cos(\log \sqrt{2})) \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}
\]

which equals

\[
(\sqrt{2} \cos(1) \cos(\log \sqrt{2}), 0).
\]
The question just asks for \( \frac{\partial (f \circ T)}{\partial s}(1,0) \), which is the first entry of \( D(T \circ s)(1,0) \), ie.
\[
\sqrt{2} \cos(1) \cos(\log \sqrt{2}).
\]

(b) \# 15.

The question is asking for
\[
D(f \circ c)(0)
\]

By the chain rule, this equals
\[
Df(c(0))Dc(0) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x-y} & -e^{x-y} \end{bmatrix} \bigg|_{(0,0)} c'(0).
\]
This in turn equals
\[
\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}
\]

(c) \# 24.

The problem is wrong to use \( \frac{\partial w}{\partial x} \) for two different things. The symbol \( \frac{\partial w}{\partial x} \) on the left hand side denotes the derivative of \( f(x, y, g(x, y)) \) with respect to \( x \), while on the right hand side it denotes the derivative of \( f(x, y, z) \) with respect to \( x \), treating \( z \) as a constant. These two things are not the same in general (try it with some nonconstant function \( g \) and some \( f \)). This is what causes the problem.

(3) Section 2.6 p.171-173

(a) \# 4(b).

The surface is given by \( g(x, y, z) = 0 \) where \( g(x, y, z) = y^2 - x^2 - 3 \). Then \( \nabla g = (-2x, 2y, 0) \). The equation of the tangent plane is therefore
\[
(-2, 4, 0) \cdot (x - 1, y - 2, z - 8) = 0.
\]

(b) \# 14(a).

The directional derivative is \( \nabla f \cdot v \) (there is no need to normalize \( v \) since it is already a unit vector).
\[
\nabla f = (y^2 + z^3, 2xy + 2yz^3, 3y^2z^2 + 3z^2x)
\]
at $(4, -2, -1)$, this has the value $(3, -12, 24)$. The answer is therefore $3/\sqrt{14} - 36/\sqrt{14} + 48/\sqrt{14}$.

(c) # 15.

(4) The surface of a mountain is given by the set of points $(x, y, z)$ in $\mathbb{R}^3$ satisfying $z = 20 - (\frac{x}{10})^2 - (\frac{y}{20})^4$ and $x, y, z \geq 0$. Klaus is at the point $(10, 20, 18)$ and he wants to toboggan down the mountain in the steepest direction possible. In which direction should he go?

Let $z = f(x, y) = 20 - (\frac{x}{10})^2 - (\frac{y}{20})^4$. Then the direction of steepest descent is $-\nabla f(10, 20) = -(-\frac{1}{10}, 2\frac{10}{10}, -\frac{1}{20}(\frac{20}{20})^3) = (\frac{1}{5}, \frac{1}{5})$. In other words, Klaus should go due northeast.

(5) A $2 \times 2$ matrix of real numbers

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
$$

can be identified with the point $(a, b, c, d) \in \mathbb{R}^4$. We can therefore define a map $f : \mathbb{R}^4 \to \mathbb{R}^4$ by $f(A) = A^2$, the square of the matrix $A$. Calculate the derivative of $f$ at an arbitrary point $B \in \mathbb{R}^4$ (also regarded as a matrix).

Write $f$ as

$$f(a, b, c, d) = (a^2 + bc, ab + bd, ac + cd, cb + d^2).$$

Then

$$Df(a, b, c, d) =
\begin{bmatrix}
2a & c & b & 0 \\
b & a + d & 0 & b \\
c & 0 & a + d & c \\
0 & c & b & 2d
\end{bmatrix}$$