Math 4320 : Introduction to Algebra
Prelim II (Chapter 2 & 3)
(due April 17, 2009, 1:25pm)

You are not allowed to discuss your answers with others. Books, lecture notes and calculators are allowed.

Part I. Group theory

1. (2.88) Show that a finite group $G$ generated by two elements of order 2 is isomorphic to a dihedral group $D_{2n}$ for some $n$.

2. Let $G$ be a group of order $n$, and let $F$ be any field. Prove that $G$ is isomorphic to a subgroup of $GL_n(F)$.

3. Rule out as many of the followings as possible as Class Equations for a group of order 10:

$$3 + 2 + 5, 1 + 2 + 2 + 5, 1 + 2 + 3 + 4, 2 + 2 + 2 + 2 + 2.$$ 

(Note that the first term in each expression corresponds to the center $Z(G)$ in $|G| = |Z(G)| + \sum |G : C_G(x)|$.)

4. Determine the class equation for each of the following groups.

   (1) $D_6$, (2) $D_{10}$, (3) $D_{2n}$

   (4) the group of upper triangular matrices in $GL_2(\mathbb{F}_3)$

(You may choose not to write down part (1) and part (2) if you know part (3).)

5. Show that $A_n$ is a simple group for all $n \geq 5$ by showing Exercise 2.127.

6. Determine all finite groups which contain at most three conjugacy classes.
Part II. Rings and fields

The following set of problems is to show that the ring

\[ R = \mathbb{Z}[\theta] = \{ a + b\theta : a, b \in \mathbb{Z} \}, \]

where \( \theta = \frac{1 + \sqrt{-19}}{2} \), is a principal ideal domain (PID) that is not a Euclidean domain (ED) (a result of Motzkin).

7. Let \( F = \{ a + b\sqrt{-19} : a, b \in \mathbb{Q} \} \subset \mathbb{C} \).

(a) Show that \( R \) is a ring, \( R \subset F \) and \( F \) is a field. Conclude that \( R \) is an integral domain. Show that \( F \) is the field of fractions of \( R \).

(b) Define \( N(a + b\sqrt{-19}) = a^2 + 19b^2 \). Prove that \( N(\alpha) > 0 \) for \( \alpha \in F - \{0\} \), and that \( N \) is multiplicative, i.e. \( N(\alpha\beta) = N(\alpha)N(\beta) \). Also prove that \( N(\alpha) \) is a positive integer for every \( \alpha \in R \).

(c) Prove that ±1 are the only units in \( R \).

8. (Criterion of Dedekind and Hasse) Let \( S \) be an integral domain and let \( N \) denote any function from \( S \) to \( \mathbb{Z} \) which satisfies \( N(\alpha) > 0 \) for \( \alpha \neq 0 \). Suppose that for every \( \alpha, \beta \in S \) with \( N(\alpha) \geq N(\beta) \), either \( \beta \) divides \( \alpha \) in \( S \), or there exist \( s, t \in S \) with \( 0 < N(s\alpha - t\beta) < N(\beta) \). (\( \ast \))

Show that \( S \) is a PID. (Hint : Let \( I \) be a non-zero ideal in \( S \) and let \( \beta \) be a non-zero element of \( I \) with \( N(\beta) \) minimal. If \( \alpha \in I \), then \( s\alpha - t\beta \) is also in \( I \) for all \( s, t \in S \). Use minimality of \( \beta \).)

9. (\( R \) is a PID) Show that the ring \( R \), with the function \( N \) defined in problem 7 satisfies the criterion of Dedekind and Hasse. (Hint : Since \( N \) is multiplicative, the condition (\( \ast \)) is equivalent to

\[ 0 < N\left(\frac{\alpha}{\beta}s - t\right) < 1. \]

Suppose \( \beta \nmid \alpha \). Write \( \frac{\alpha}{\beta} = \frac{a + b\sqrt{-19}}{c} \) in \( F \) with integers \( a, b, c \) having no common divisor and with \( c > 1 \). Divide into four cases, \( c \geq 5 \) and \( c = 2, 3, 4 \).

10. (\( R \) is not a ED) Let \( D \) be an integral domain. Recall that a non-zero non-unit element \( u \in D \) is called a universal side divisor if for every \( x \in D \), there is some unit \( z \in D \) such that \( u \) divides \( x - z \) in \( D \). Prove that the ring \( R \) above has no universal side divisors, hence is not a ED. (Hint : Use part (c) of problem 7. Use first \( x = 2 \), then \( x = \theta \). What are the divisors of 2, 3 in \( R \)?)