Math 2940 Solutions, Fall 2012
Section 6.7

3) \( p(-1)q(-1) + p(0)q(0) + p(1)q(1) = 3 + 20 + 5 = 28. \)

4) \( p(-1)q(-1) + p(0)q(0) + p(1)q(1) = -20 + 0 + 10 = -10. \)

21) \( \int_0^1 (1 - 3t^2)(t - t^3) dt = \int_0^1 t - 4t^3 + 3t^5 dt = (t^2/2 - t^4 + t^6/2)|_0^1 = 1/2 - 1 + 1/2 = 0. \)

23) \( ||f||^2 = \int_0^1 (1 - 3t^2)^2 dt = \int_0^1 1 - 6t^2 + 9t^4 dt = (t - 2t^3 + 1.8t^5)|_0^1 = 1 - 2 + 1.8 = .8 \) so \( ||f|| = \sqrt{.8}. \)

25) We have to do Gram-Schmidt with respect to this basis. Note \( <1, t> = \int_{-1}^1 1 \cdot t dt = (t^2/2)|_{-1}^1 = 0. \) We’re in luck! The first two vectors are orthogonal. Note also \( <1, 1> = \int_{-1}^1 1 dt = 2, <1, 1> = \int_{-1}^1 1 dt = 2, <t^2, 1> = \int_{-1}^1 t^2 dt = t^3/3|_{-1}^1 = 2/3, <t, t> = \int_{-1}^1 t^2 dt = t^3/3|_{-1}^1 = 2/3 \) and \( <t^2, t> = \int_{-1}^1 t^3 dt = t^4/4|_{-1}^1 = 0. \) GS tells us the third in our basis is

\[
t^2 - \frac{<t^2, 1>}{<1, 1>} \cdot 1 - \frac{<t^2, t>}{<t, t>} t = t^2 - \frac{2/3}{2} \cdot 1 - \frac{0}{2/3} t = t^2 - (1/3).
\]

So \( \{1, t, t^2 - (1/3)\} \) forms an orthogonal basis.

28) I use \( x \) here instead of \( t \) because I forgot. Set \( I_j = \int_0^{2\pi} \cos^j x \, dx. \) Then \( I_0 = 2\pi, I_1 = I_3 = I_5 = 0 \) and \( I_2 = \pi \) and \( I_4 = 3\pi/4. \) Note \( <\cos x, \cos^s x> = I_{r+s} \) which is 0 for \( r + s \) odd. Thus \( <\cos x, \cos x> = 0 \) and our first two vectors are orthogonal. The third vector of the basis is just \( \cos^2 x - \frac{I_2}{I_0} \cdot 1 - \frac{I_3}{I_2} \cos t = \cos^2 x - (1/2). \) The fourth vector of our basis is \( \cos^3 x - \frac{I_3}{I_0} \cdot 1 - \frac{I_4}{I_2} \cos x = \frac{<\cos^3 x, \cos^2 x - (1/2)>}{<\cos x - (1/2), \cos^2 x - (1/2)>} (\cos^2 x - (1/2)) = \cos^3 x - (3/4) \cos x \) so our basis is \( \{1, \cos x, \cos^2 x - (1/2), \cos^3 x - (3/4) \cos x\}. \)