4.5

4) The set of all vectors \[
\begin{bmatrix}
p + 2q \\
-p \\
3p - q \\
p + q
\end{bmatrix}
\] is the span of the two vectors \[
\begin{bmatrix}
1 \\
-1 \\
3 \\
1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
2 \\
0 \\
-1 \\
1
\end{bmatrix}.
\]

These are clearly not multiples of one another, so their span is two dimensional, and they form a basis of the span.

12) We are asked to find the dimension of the span of the four vectors
\[
\begin{bmatrix}
-2 \\
3 \\
5
\end{bmatrix}
\text{ and }
\begin{bmatrix}
-3 \\
5 \\
5
\end{bmatrix}.
\]
We’ve done a zillion row reductions, so let’s talk this one out. The first two vectors are not multiples of one another, so they are independent. The span of these first two vectors only includes vectors whose third entry is 0. So \[
\begin{bmatrix}
-2 \\
3 \\
5
\end{bmatrix}
\] is NOT in the span of the first two vectors. Thus the span of the first three vectors is three dimensional. As we are working in \(\mathbb{R}^3\), the span of all four vectors in three dimensional.

13) \(A =
\begin{bmatrix}
1 & -6 & 9 & 0 & 2 \\
0 & 1 & 2 & -4 & 5 \\
0 & 0 & 0 & 5 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] is in echelon form The first, second and fourth columns are pivotal so \(\text{Col } A\) is three dimensional. Then \(\text{Nul } A\) has dimension \(5 - 3 = 2\).

14) \(A =
\begin{bmatrix}
1 & 2 & -4 & 3 & -2 & 6 & 0 \\
0 & 0 & 0 & 1 & 0 & -3 & 7 \\
0 & 0 & 0 & 0 & 1 & 4 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] is in echelon form The first, fourth, fifth and seventh columns are pivotal so \(\text{Col } A\) is four dimensional. Then \(\text{Nul } A\) has dimension \(7 - 4 = 3\).

20) (a) FALSE. Ok, this is kind of a technicality. The problem is \(\mathbb{R}^2\) is not contained in \(\mathbb{R}^3\), one being ordered triples of real numbers, the other being ordered pairs. The xy-plane consisting of elements of \(\mathbb{R}^3\) of the form \[
\begin{bmatrix}
x \\
y \\
0
\end{bmatrix}
\] is a two dimensional subspace of \(\mathbb{R}^3\).

(b) FALSE. \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = \vec{0}
\] has two variables and \(\text{Nul } A\) is zero dimensional.

(c) FALSE. \(\mathbb{R}^2\) is spanned by the infinite collection of vectors \((\alpha, 1/\alpha)\) as \(\alpha\) runs through all
nonzero reals, but \( \mathbb{R}^2 \) is two dimensional.

(d) FALSE. This is only true if \( S \) has \( n \) elements. It will always have at least \( n \) elements, but it could have more. See (c).

(e) TRUE. Take a basis of the given three dimensional subspace. Make a matrix \( A \) out of these vectors. Then \( A \) has three pivots upon row reduction, so \( Ax = \vec{b} \) is solvable for all \( \vec{b} \in \mathbb{R}^3 \), that is the columns of \( A \) span \( \mathbb{R}^3 \).

21) We look at the coordinate matrix for the Hermite polynomials in terms of the standard basis of \( \mathbb{P}_3 \), \( \{1, t, t^2, t^3\} \). This is

\[
\begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 2 & 0 & -12 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 8
\end{bmatrix}
\]

There are 4 pivots so these vectors are independent and span a 4 dimensional subspace of \( \mathbb{P}_3 \). Since \( \mathbb{P}_3 \) is four dimensional they span \( \mathbb{P}_3 \).

My problems

1) This one is tricky. Before getting to it, let’s think about an easier case, when \( X \) and \( Y \) are two dimensional subspaces of \( \mathbb{R}^3 \). So we have two planes through the origin. They could be the same, or they could intersect in a line. Note their intersection is never just the origin. So the intersection is 1 or 2 dimensional, but not 0 dimensional. Now think about the following 2 dimensional subspaces of \( \mathbb{R}^4 \), the subspace \( X \) consisting of all points of the form \((x, y, 0, 0)\) and the subspace \( Y \) consisting of all points of the form \((0, 0, z, w)\). What is the intersection of these two planes? Any point in the intersection has to have both its first two coordinates 0 and its last two coordinates 0. The intersection of \( X \) and \( Y \) is just the origin, 0 dimensional! So in \( \mathbb{R}^3 \) there is not enough room for a couple 2 dimensional subspaces to intersect at just the origin, but in \( \mathbb{R}^4 \) there is.

Another way to think about this is via equations. Our planes in \( \mathbb{R}^3 \) are the null spaces of one equation, so their intersection is the null space of a system of two equations. There will be at most two pivots and thus at least \( 3 - 2 = 1 \) free variable. So the intersection is at least one dimensional. In \( \mathbb{R}^4 \) our planes are the null space of two equations each, so the intersection is the null space of a system of 4 equations. There could be as many as 4 pivots and as few as \( 4 - 4 = 0 \) free variables, so the intersection could be 0 dimensional.

On to a basis oriented proof of the problem assigned: If \( X = Y \) then the intersection is 5 dimensional. Henceforth suppose \( X \neq Y \).

Suppose the intersection of \( X \) and \( Y \) is \( d \) dimensional with basis \( \{ \vec{z}_1, ..., \vec{z}_d \} \). Since \( X \neq Y \) we know \( d < 5 \). Choose vectors \( \vec{x}_{d+1}, ..., \vec{x}_5 \) such that \( \{ \vec{z}_1, ..., \vec{z}_d, \vec{x}_{d+1}, ..., \vec{x}_5 \} \) form a basis of \( X \) and vectors \( \vec{y}_{d+1}, ..., \vec{y}_5 \) such that \( \{ \vec{z}_1, ..., \vec{z}_d, \vec{y}_{d+1}, ..., \vec{y}_5 \} \) form a basis of \( Y \). I claim \( \{ \vec{z}_1, ..., \vec{v}_d, \vec{x}_{d+1}, ..., \vec{x}_5, \vec{y}_{d+1}, ..., \vec{y}_5 \} \) is independent. In what follows, LC denotes ‘some linear combination of’. Then we start with

\[
\text{LC}(\vec{z}_i, \vec{x}_i, \vec{y}_i) = 0
\]

and we will show that all the coefficients in the LCs are 0. That is enough. We rewrite this as

\[
(\ast) \quad \text{LC}(\vec{z}_i, \vec{x}_i) = -\text{LC}(\vec{y}_i).
\]
The right hand side of (\(\ast\)) is then in \(Y\), so the left side is in \(Y\). But the left hand side is a combination of the \(\vec{z}_i\) and \(\vec{x}_i\) so it is in \(X\). Thus both sides are in both \(X\) and \(Y\) and hence spanned by the \(\vec{z}_i\). So equating the left side with an LC of the \(\vec{z}_i\), we have

\[
(**) \quad \text{LC}(\vec{z}_i, \vec{x}_i) = -\text{LC}(\vec{z}_i).
\]

As the \(\vec{z}_i\) and \(\vec{x}_i\) form a basis of \(X\), all coefficients on both sides of (\(\ast\ast\)) are 0. So we have taken care of the left side of (\(\ast\)), which now becomes \(\vec{0} = -\text{LC}(\vec{y}_i)\). As the \(\vec{y}_i\) are independent (being part of a basis of \(Y\)), the coefficients on the right side are 0 and we are done.

Done with what? Well, the \(\vec{z}_i, \vec{x}_i, \vec{y}_i\) are independent. There are

\[
d + (5 - d) + (5 - d) = 10 - d
\]

vectors all together. If we are in \(\mathbb{R}^6\) we can have at most 6 independent vectors so \(10 - d \leq 6\) so \(d \geq 4\). We already know \(d \leq 5\). An example of \(d = 4\) is as follows: Let

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

be a basis of \(X\) and

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

be a basis of \(Y\).

If we are in \(\mathbb{R}^7\), then \(10 - d \geq 7\) so \(d \geq 3\). Again, we’ve dealt with \(d = 5\). Examples where \(d = 3, 4\) are possible to produce as above. Do it.

Alternatively, in the \(\mathbb{R}^6\) case \(X\) and \(Y\) are defined by one equation each, so their intersection is defined by two equations with at most 2 pivots and at least \(6 - 2 = 4\) free variables, so the common null space of the equations (the intersection of \(X\) and \(Y\)) is at least 4 dimensional.

In the \(\mathbb{R}^7\) case \(X\) and \(Y\) are defined by two equation each, so their intersection is defined by four equations with at most 4 pivots and at least \(7 - 4 = 3\) free variables, so the common null space of the equations (the intersection of \(X\) and \(Y\)) is at least 3 dimensional.