Math 2940 Solutions, Fall 2012
Section 3.2 and 3.3

3.2

8) \[
\begin{vmatrix}
1 & 3 & 3 & -4 \\
0 & 1 & 2 & -5 \\
2 & 5 & 4 & -3 \\
-3 & -7 & -5 & 2
\end{vmatrix}
= \begin{vmatrix}
1 & 3 & 3 & -4 \\
0 & 1 & 2 & -5 \\
0 & -1 & -2 & 5 \\
0 & 0 & 0 & 0
\end{vmatrix} = 0.
\]

19) \[
\begin{vmatrix}
a & b & c \\
2d + a & 2e + b & 2f + c \\
g & h & i
\end{vmatrix}
= \begin{vmatrix}
a & b & c \\
2d & 2e & 2f \\
g & h & i
\end{vmatrix}
= \begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{vmatrix} = 14.
\]

22) We compute \( d = \begin{vmatrix}
5 & 0 & -1 \\
1 & -3 & -2 \\
0 & 5 & 3
\end{vmatrix} \) by expanding along the first row to get

\[ d = 5 \begin{vmatrix}
-3 & -2 & -1 \\
5 & 3 & 0 \\
0 & -1 & 5
\end{vmatrix} = 5(-9 + 10) - 0 + (-1)(5 - 0) = 0. \]

The matrix is not invertible.

24) Expanding on the second column

\[ \begin{vmatrix}
4 & -7 & -3 \\
6 & 0 & -5 \\
-7 & 2 & 6
\end{vmatrix}
= -7 \begin{vmatrix}
6 & -5 \\
-7 & 6
\end{vmatrix} + 0\begin{vmatrix}
4 & -3 \\
6 & -5
\end{vmatrix} = 7(36 - 35) - 2(-20 - 18) = 7 + 4 = 11. \]

The vectors are independent.

29) Since \( \det(XY) = \det(X)\det(Y) \), iterating with \( X = Y = b \) gives \( \det(B^5) = \det(B)^5 \).

Now expanding along the first row,

\[ |B| = \begin{vmatrix}
1 & 0 & 1 \\
1 & 1 & 2 \\
1 & 2 & 1
\end{vmatrix} = 1 \begin{vmatrix}
2 & 1 \\
1 & 2
\end{vmatrix} - 0 + 1 \begin{vmatrix}
1 & 1 \\
1 & 2
\end{vmatrix} = 1(4 - 0) - 0 + 1(2 - 1) = -2. \]

So \( \det(B^5) = (-2)^5 = -32. \)

45) Say \( A \) is \( m \times n \) where \( n > m \), that is \( A \) has more columns than rows. Then \( AA^T \) is \( m \times m \) and \( A^T A \) is \( n \times n \). What you find with examples is that \( \det(AA^T) \) takes on a variety of values, usually not 0, but \( \det(A^T A) \) is always 0. Why is this? Well \( \det(A^T A) = 0 \) exactly when the columns of \( A^T A \) are dependent. If there were \( n \) independent in \( A^T A \), then there would be \( n \) independent columns in \( A^T \) (the columns of \( A^T A \) being linear combinations of the columns of \( A^T \)) but \( A^T \) has only \( m \) columns, so this cannot be the case.
On the other hand, the $m$ columns of $AA^T$ are linear combinations of the $n > m$ columns of $A$. For a random $A$ these columns will be independent, as will their corresponding linear combinations.

### 3.3

2) \[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}.
\]

7) \[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \frac{1}{12s^2-36} \begin{bmatrix} 2s & -4 \\ -9 & 6s \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \frac{1}{12s^2-36} \begin{bmatrix} 10s + 8 \\ -12s - 45 \end{bmatrix}.
\]
As long as $12s^2 \neq 36$, there is exactly one solution. When $12s^2 = 36$, that is $s = \pm \sqrt{3}$ we have that $A$ is not invertible. In those situations, there will be either no solutions or infinitely many.

I haven’t worked out what happens in those cases. There are probably no solutions. Why?

18) Well \[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \]
So if $|A| = ad - bc = 1$ and $a, b, c$ and $d$ are integers, then $A^{-1}$ has integer entries too.

What about the $n \times n$ case? Well, $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$ where $\text{adj}(A)$ has integer entries because $A$ does. Then since $\det(A) = 1$, $A^{-1}$ has integer entries.

24) The volume in question is just \[
d = \begin{vmatrix} 1 & -2 & -1 \\ 4 & -5 & 2 \\ 0 & 2 & -1 \end{vmatrix}.
\]
Expanding on column 1,

\[
d = 1 \begin{vmatrix} -5 & 2 \\ 2 & -1 \end{vmatrix} - 4 \begin{vmatrix} -2 & -1 \\ 2 & -1 \end{vmatrix} = 1 - 4(4) = -15.
\]
So the volume is 15.

31) (a) $T$ scales by a factor of $a$ in the $x_1$ direction, $b$ in the $x_2$ direction and $c$ in the $x_3$ direction. Before applying $T$, we have $x_1^2 + x_2^2 + x_3^2 = 1$. After the scaling, $|x_1|$ maxes out at $a$, $|x_2|$ at $b$ and $|x_3|$ at $c$, so the equation $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ is satisfied.

(b) The volume of the sphere is $4\pi/3$, so the volume of the ellipsoid is just scaled by $a$, $b$ and $c$ in the $x_1$, $x_2$ and $x_3$ directions. So it is $4\pi abc/3$.

### My problems

1) Say an $n - 1 \times n - 1$ determinant takes $f(n - 1)$ computations. Then to compute an $n \times n$ determinant by cofactors involves computing $n$ of these determinants along with $n$
multiplications and \( n - 1 \) additions. So \( f(n) = nf(n-1) + n > nf(n-1) \). Since \( f(1) = 1 \), we have \( f(n) \geq nf(n-1) \geq n(n-1)f(n-2) \geq \ldots \geq n! \). The \( n \) we added on will be small relative to this main term. Now \( 10! = 3628800 \), just a few million, but \( 20! \) is roughly \( 2.4 \times 10^{18} \). On a gigahertz machine, this would take more than \( 2.4 \times 10^9 \) seconds. That’s over 60 years. Observe \( 15! \) is roughly \( 1.3 \times 10^{12} \). This would take \( 1.3 \times 10^3 \) seconds. That’s manageable, but any bigger is getting to be a problem.