1.10

11) (a) The migration rate from California is $516100/31524000 = .016372$. The fraction of Californians who stayed is $1 - .016372 = .983628$. To five places these are .01637 and .98363.

The migration rate to California is $381262/228680000 = .00167$ the fraction of outsiders who remained away from California is $2 - .00167 = .99833$. Our matrix is $A = \begin{bmatrix} .98363 & .00167 \\ .01637 & .99833 \end{bmatrix}$.

(b) We have initial populations (in millions) in and out of California given by $\vec{x}_{1994} = \begin{bmatrix} 31.524 \\ 228.68 \end{bmatrix}$. For populations in 2000 we see $x_{2000} = A^6 x_{1994}$ which a program computes to be $\begin{bmatrix} 30.755 \\ 229.45 \end{bmatrix}$.

Note this model is very unrealistic. It incorporates no births or deaths.

12) Setting $A = \begin{bmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 295 \\ 55 \\ 150 \end{bmatrix}$, we need $A^2 \vec{x}_0$ which is $\begin{bmatrix} 311.54 \\ 58.255 \\ 130.202 \end{bmatrix}$.

Of course we should round this to $\begin{bmatrix} 312 \\ 58 \\ 130 \end{bmatrix}$.

My problems

1) The rotation matrix is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ while the stretching matrix is $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. If we first stretch $\vec{x}$ we get $B\vec{x}$. If we rotate this output, we have to find $A(B\vec{x}) = (AB)\vec{x}$ so we need $AB$ which is $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$.

2) This is the sort of thing you need to understand well if you are going to make good video games. To understand a linear transformation, we need to know what it does to $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$. Go get a Rubik’s or other cube, hold it with faces parallel to the walls and hold opposite corners fixed and rotate it until the faces are again parallel to the wall. If you do that three times it will have returned to the original position. So one such rotation is $2\pi/3$ radians. In the orientation I have in mind, $\vec{e}_i$ goes to $\vec{e}_{i+1}$ where we understand that $\vec{e}_4$ really means $\vec{e}_1$.

Our matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The other correct answer (rotating the other way) is the inverse
of this matrix), \[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}.
\]

3) We need to compute \(A^m\vec{x}_0\) for large \(m\). So \(A^{10}\vec{x}_0 = \begin{bmatrix} 345.57 \\ 56.36 \\ 98.05 \end{bmatrix}\). Also \(A^{20}\vec{x}_0 = \begin{bmatrix} 359.74 \\ 49.896 \\ 90.362 \end{bmatrix}\).

\[
A^{30}\vec{x}_0 = \begin{bmatrix} 364.63 \\ 46.507 \\ 88.863 \end{bmatrix}, \quad A^{40}\vec{x}_0 = \begin{bmatrix} 366.47 \\ 45.072 \\ 88.457 \end{bmatrix}, \quad A^{100}\vec{x}_0 = \begin{bmatrix} 367.64 \\ 44.121 \\ 88.236 \end{bmatrix} \quad \text{and} \quad A^{10000}\vec{x}_0 = \begin{bmatrix} 367.65 \\ 44.118 \\ 88.235 \end{bmatrix}.
\]

We seem to have reached a long term equilibrium. Stay tuned for more!