1 Solving Linear Recurrences

Suppose we have a sequence \( a_n \) defined by the formula

\[
a_n = 3a_{n-1} + 1, \quad a_0 = 1
\]

The first couple terms of the sequence are

\[1, \quad 4, \quad 13, \quad 40, \quad 121, \quad \ldots\]

Can you find a general closed for expression for the \( n \text{th} \) term of this sequence?

One big method for dealing with problems of this sort is the method of \textit{generating functions}. Generating functions let us associate a function to a sequence of numbers. Then, if we are lucky, properties of the sequence of numbers can be ascertained by analyzing the corresponding function.

The simplest kind of generating function is called an \textit{ordinary generating function}. Given a sequence \( a_n \), the corresponding OGF is the function

\[a(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots = \sum_{n=0}^{\infty} a_n z^n\]

\textbf{Example 1a}: The OGF for the constant sequence \( c_n = 1 \) is the function

\[c(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1 - z}\]

\textbf{Example 1b}: The OGF for the sequence

\[E_n = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}
\]

is the function

\[E(z) = 1 + z^2 + z^4 + \cdots = \frac{1}{1 - z^2}\]

\textbf{Example 1c}: The OGF for the sequence

\[O_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}
\]

is the function

\[O(z) = z + z^3 + z^5 + \cdots = \frac{z}{1 - z^2}\]
Example 1d: Fix a number $k$. The OGF for the sequence

$$K_n = k^n$$

is the function

$$K(z) = 1 + kz + k^2z^2 + k^3z^3 + \cdots = \frac{1}{1-kz}$$

Example 1e: The OGF for the sequence $I_n = n$ can be computed from what we already know. First, note that

$$I(z) = 0 + 1 \cdot z + 2 \cdot z^2 + 3 \cdot z^3 + \cdots = z \cdot (1 + 2z + 3z^2 + 4z^3 + \ldots)$$

The series is parenthesis is just the derivative of the function $\frac{1}{1-z}$. So altogether, we have

$$I(z) = z \cdot \frac{d}{dz} \left( \frac{1}{1-z} \right) = \frac{z}{(1-z)^2}$$

Now, let us return to the recurrence and see how generating functions can help us solve it. To begin, let us write the recurrence in an unusual way. The recurrence relation essentially claims that each one of the following sums is correct:

$$3a_0 + 3a_1 + 3a_2 + 3a_3 + \cdots + 1 + 1 + 1 + 1 + 1 + \cdots$$

$$\frac{3a_0z}{a_0} + \frac{3a_1z^2}{a_1} + \frac{3a_2z^3}{a_2} + \frac{3a_3z^4}{a_3} + \frac{3a_4z^5}{a_4} + \cdots$$

Or, equivalently,

$$3z \cdot a(z) + \frac{1}{1-z} = a(z)$$

It is simple enough to solve this equation for $a(z)$, giving

$$a(z) = \frac{1}{(1-z)(1-3z)}$$

So, in some sense, we now have a formula for the $n^{th}$ term $a_n$: it is just the $n^{th}$ coefficient in the power series of $\frac{1}{(1-z)(1-3z)}$. In other words,

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n} \left( \frac{1}{(1-z)(1-3z)} \right) \bigg|_{z=0}$$

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This is interesting, but maybe not so useful. After all, we probably don’t want to compute 100 derivatives just to find $a_{100}$.

However, we can do even more. If you recall the method of partial fractions, you will remember that we can write

$$\frac{1}{(1 - z)(1 - 3z)} = \frac{A}{1 - z} + \frac{B}{1 - 3z}$$

for some numbers $A$ and $B$. Cross multiplying, we find that $A$ and $B$ must satisfy

$$1 = (A + B) - z \cdot (3A + B)$$

so that $B = -3A$ and $A + B = 1$. Thus, $A = -1/2$ and $B = 3/2$.

Altogether, we have now shown that

$$a(z) = \frac{3/2}{1 - 3z} - \frac{1/2}{1 - z}$$

But both of these functions are very simple! The $n^{th}$ coefficient of $\frac{1}{1 - z}$ is just 1, while the $n^{th}$ coefficient of $\frac{1}{1 - 3z}$ is just $3^n$. So altogether, the $n^{th}$ coefficient of $a(z)$ must be

$$a_n = \frac{3}{2} \cdot 3^n - \frac{1}{2} = \frac{1}{2}(3^{n+1} - 1)$$

Thus, for example, $a_3 = \frac{1}{2}(81 - 1) = 40$ as previously computed. And we can easily compute $a_{100} = (3^{101} - 1)/2$.

## 2 Manipulating Generating Functions

Let us write $[f]_n$ for the coefficient of $z^n$ in $f(z)$. Then it is easy to verify the following facts:

1. If $f(z)$ is the generating function for the sequence $f_n$, then $[f]_n = f_n$.
2. If $c$ is a number, then $[cf]_n = c[f]_n$.
3. $[f + g]_n = [f]_n + [g]_n$.
4. $[zf]_n = [f]_{n-1}$. Thus, multiplying by $z$ corresponds to the **shift operator**.
5. $[\frac{d}{dz}f]_n = (n + 1)[f]_{n+1}$.
6. Combining the previous two, $[z \frac{d}{dz}f]_n = n[f]_n$. Because of this, $z \frac{d}{dz}$ is called the **number operator**. Note that we used the number operator to find the generating function of $I_n = n$ from the generating function $c_n = 1$. 


Problem: Using these rules and series which you already know, find the generating function for the following sequences:

\[
\begin{align*}
  a_n &= (-1)^n \\
  b_n &= n + 1 \\
  c_n &= 2n \\
  d_n &= n^2 \\
  e_n &= 1/n! \\
  f_n &= 1/(n + 1) \\
  g_n &= \binom{m}{n}
\end{align*}
\]

Problem: Let \( f, g \) be functions. Find a formula to compute \([fg]_n\).

Problem: Use generating functions to re-derive the identity

\[
\sum_{k=0}^{n} k = \frac{n(n+1)}{2}
\]

(Hint: if we call the sum on the right \( a_n \), then we have the recurrence \( a_n = a_{n-1} + n \).)

Use a similar technique to find a formula for \( \sum k^2 \).

3 The Fibonacci Numbers

As another extended example, let us consider the Fibonacci sequence \( F_n \). This sequence has the famous property that

\[
F_n = F_{n-1} + F_{n-2}
\]

with \( F_0 = F_1 = 1 \). So we have an equation

\[
\begin{array}{cccccc}
  & F_0 & F_1 & F_2 & \ldots \\
  + & F_0 & F_1 & F_2 & F_3 & \ldots \\
  + & 1 & F_0 & F_1 & F_2 & F_3 & \ldots \\
\end{array}
\]

Inserting powers of \( z \) to move into generating function land, this sum is equivalent to the statement

\[
z^2 F(z) + z F(z) + 1 = F(z)
\]
Solving for $F(z)$, we get

$$F(z) = \frac{1}{1 - z - z^2}$$

Let us use partial fractions to break this function up and (hopefully) find a simple expression for $F_n$.

The roots of $1 - z - z^2$ are

$$-\Phi = -\frac{1}{2} - \frac{\sqrt{5}}{2}, \quad \phi = \frac{\sqrt{5}}{2} - \frac{1}{2}$$

Note for later that $\Phi \cdot \phi = 1$ and $\Phi + \phi = \sqrt{5}$. We want a partial fraction expansion of the form

$$\frac{1}{1 - z - z^2} = \frac{A}{z + \Phi} + \frac{B}{\phi - z}$$

So we have $A(\phi - z) + B(z + \Phi) = 1$, which implies that $A = B$ and $A = (\phi + \Phi)^{-1} = 1/\sqrt{5}$. Thus, we have a partial fractions decomposition of the Fibonacci generating function:

$$F(z) = \frac{1}{\sqrt{5}} \left( \frac{1}{z + \Phi} + \frac{1}{\phi - z} \right)$$

Let us manipulate the inner fractions a bit so that they look like the sums of geometric series:

$$F(z) = \frac{1}{\sqrt{5}} \left( \frac{1/\Phi}{1 + z/\Phi} + \frac{1/\phi}{1 - z/\phi} \right)$$

Altogether, this gives us the surprising closed form expression

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1}{\phi^{n+1}} + (-1)^n \frac{1}{\phi^{n+1}} \right) = \frac{1}{\sqrt{5}} \left( \Phi^{n+1} + (-1)^n \phi^{n+1} \right)$$

In fact, since $|\phi| < 1$ we get the very simple yet accurate approximation

$$F_n \sim \frac{\Phi^{n+1}}{\sqrt{5}}$$

For example, $\Phi_{50}/\sqrt{5} = 12586269025.000057$, and $F_{49} = 12586269025$. From this, we can also immediately see the famous relationship between ratios of Fibonacci numbers and the golden ratio, namely

$$\lim_{n \to \infty} F_n/F_{n-1} = \Phi$$
4 Constructing Recurrence Relations

**Problem:** Let $f_n$ be the number of ways to partition a row of $n$ dots into ones and twos. For example, $f_4 = 5$ because $\circ \circ \circ \circ$ can be partitioned as follows:

\[
\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

Find the generating function $f(z)$.

**Problem:** Let $p_n$ be the number of $n$-step paths in the plane, starting at $(0,0)$, where each step is either of the form $(1,0)$, $(0,1)$ or $(0,-1)$ and a path is not allowed to intersect itself.

1. Find a recurrence relation satisfied by the $p_n$. Hint: you should be able to find a relation between $p_n$, $p_{n-1}$ and $p_{n-2}$.
2. Starting from the recurrence relation, find the generating function $p(z)$.
3. Using the method of partial fractions on $p(z)$, find an explicit expression for $p_n$.
4. What is the probability that a random path of length $n$, using the steps $(1,0)$, $(0,1)$, or $(0,-1)$, does not intersect itself?

**Problem:** Suppose that $f_n$ is a sequence which satisfies the linear recurrence

\[
c_0 f_n + c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_k f_{n-k}
\]

where the $c_i$ are constants.

1. Show that the generating function $f(z)$ is a rational function with denominator

\[
Q(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_k z^k
\]

Show that the degree of the numerator is less than $k$.
2. Now show the converse; if $f(z)$ is a rational function $P(z)/Q(z)$ with $\deg P < \deg Q$ then $[f]_n$ satisfies the recurrence determined by $Q$. 


Problem: Given a sequence \( a_n \), define a new sequence \( \sigma^a_n \) by

\[
\sigma^a_n = \sum_{k=0}^{n} a_n
\]

Find the generating function \( \sigma^a(z) \) in terms of \( a(z) \). Then, show that the Fibonacci numbers satisfy

\[
F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1
\]

Problem: A derangement is a permutation with no fixed points. Let \( D_n \) denote the number of derangements of \( n \) objects. Give a combinatorial proof that \( n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k} \). Interpret this identity with exponential generating functions to find the generating function \( D(z) \).