2.1.2

Suppose \( \{a_n\} , \{b_n\} \) are bounded below. Then for all \( n \)
\( A_1 \leq a_n \leq A_2 \) for some \( A_1, A_2 \) and
\( B_1 \leq b_n \leq B_2 \) for some \( B_1, B_2 \).

(i) Let \( A = \max \{ |A_1|, |A_2| \} \), \( B = \max \{ |B_1|, |B_2| \} \). Then
\[ 0 \leq |a_n| \leq A \quad \text{and} \quad 0 \leq |b_n| \leq B \quad \text{so that} \]
\[ a_nb_n \leq |a_n||b_n| \leq AB \quad \text{for all} \ n, \quad \text{so} \ \{a_nb_n\} \text{is bounded above.} \]

(ii) Alternatively, suppose one of the sequences has a limit \( L > 0 \).
Without loss of generality let \( \lim_{n \to \infty} a_n = L > 0 \).
Then for sufficiently large \( n \)
\[ \frac{a_nb_n}{|a_n||b_n|} = \frac{1}{L} \quad \text{so} \]
either \( b_n \leq 0 \) in which case \( a_nb_n \leq 0 \) or \( B > b_n > 0 \) in which case \( a_nb_n \leq LB \).

In both cases \( \{a_nb_n\} \) is bounded above for large \( n \).
Since the first \( N \) terms are bounded by \( \max \{a_nb_n|n \leq N\} \)
Then \( \{a_nb_n\} \) is bounded above.

Note: (i) can be weakened to the case \( a_n > 0, \ b_n > 0 \).
This is (still) sufficient. (ii) can be strengthened to the case \( a_n \) is eventually positive for \( n \) large.

2.2.1

(a) \( -1 \leq \cos n \leq 1 \) so \( 2 \leq 3 + \cos n \leq 4 \) and so \( \frac{1}{4} \leq \frac{1}{3+\cos n} \leq \frac{1}{2} \).

(b) Inspection suggests an upper bound (sharpest) at \( n=1 \) and
a lower bound at \( n=4 \) (again, sharpest). Indeed, for \( n \geq 2 \)
\[ -\frac{1}{n^2+1} \leq \frac{\sin n}{n^2+1} \leq \frac{1}{n^2+1} \leq \frac{1}{5} < \frac{\sin(4)}{2} \approx 0.421 \quad \text{and for} \ n \geq 5 \]
and \( -0.045 \leq \frac{\sin(4)}{17} < -\frac{1}{26} \approx -0.039 \leq \frac{1}{n^2+1} \leq \frac{\sin(n)}{n^2+1} \leq \frac{1}{n^2+1} \)

so that these are intact bounds for \( \frac{\sin(n)}{n^2+1} \).
2.4.2
Assume $|a_i| \leq 1$ for all $a_i$, then

$$|a_1 \sin(b) + a_2 \sin(2b) + \cdots + a_n \sin(nb)| \leq |a_1 \sin(b)| + |a_2 \sin(2b)| + \cdots + |a_n \sin(nb)|$$

(by extended triangle inequality)

$$\leq |a_1| + |a_2| + \cdots + |a_n|$$

$$\leq 1 + 1 + \cdots + 1$$

$$= n$$

So by contraposition, if $|a_1 \sin(b) + \cdots + a_n \sin(nb)| > n$, then $|a_i| > 1$.

2.6.1
Choose $N = a$ so that $n > N \Rightarrow$

$$\frac{a^{n+1}}{(n+1)!} = \frac{a^n}{n!} \cdot \frac{a}{(n+1)} \leq \frac{a^n}{n!}$$

since $\frac{a^n}{n!} > 0$ and $\frac{a}{n+1} < 1$.

Then the sequence is eventually monotonic (decreasing).

2.6.2
Let $b_n = \frac{n^n}{a^n}$.

Consider $b_{n+1} - b_n = \frac{n+1}{a^{n+1}} - \frac{n}{a^n}$

$$= \frac{n+1-n}{a^{n+1}}$$

$$= -\frac{n(a-1)}{a^{n+1}}$$

Since $a > 1$, then $a^n > 0$, and if $n > \frac{1}{a-1}$, $n(a-1)-1 > 0$.

Let $N = \frac{1}{a-1}$, then for any $n > N$, $b_{n+1} - b_n < 0$ i.e. $b_{n+1} < b_n$.

So, the sequence $b_n = \frac{n^n}{a^n}$ is monotonic (decreasing) for $n$ large.

Problem 2.1

(a) $a_n \leq a_{n+1}$ so

$$n(a_{n+1} \geq a_1 + \cdots + a_n)$$

so

$$n(a_1 + \cdots + a_n) + n \cdot a_{n+1} \geq n(a_1 + \cdots + a_n) + (a_1 + \cdots + a_n)$$

and

$$b_{n+1} = \frac{a_1 + \cdots + a_n + a_{n+1}}{n+1} \geq \frac{a_1 + \cdots + a_n}{n} = b_n$$

So $b_n$ is increasing.

(b) $a_n \leq A$ for some $A$ and all $n$. Then $b_n = \frac{a_1 + \cdots + a_n}{n} \leq \frac{A + \cdots + A}{n} = \frac{A}{n} = A$ for all $n$. Hence $b_n$ is bounded above.
2-2: Since \( \{a_n\} \) is bounded and increasing, then by Completeness Property, \( \{a_n\} \) has a limit \( L \). Note that \( a_n \leq L \).

Then given \( \varepsilon = \frac{1}{2} \), there exists an integer \( N > 0 \) s.t. for any \( n \geq N \).

\[ 0 \leq L - a_n \leq \frac{1}{2}. \]

Since \( L - a_n = (L - a_{n+1}) + (a_{n+1} - a_n) \geq a_{n+1} - a_n \), then \( a_{n+1} - a_n = \frac{1}{2} \) for any \( n \geq N \). Since \( \{a_n\} \) is an integer sequence, then \( a_{n+1} - a_n = 0 \) for any \( n \geq N \).